Systems & Control Letters 12 (1989) 319–326
North-Holland

New result on (f, g)-invariance *

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Received 23 April 1988
Revised 18 August and 31 December 1988

Abstract: The existing result for a nonsingular and involutive distribution Δ to be (f, g)-invariant has a priori assumption that $(\mathscr{G}+\Delta)/\Delta$ is nonsingular. This paper treats the problem of (f, g)-invariance for the case where $(\mathscr{G}+\Delta)/\Delta$ is singular. Necessary and sufficient conditions for a nonsingular and involutive distribution to be (f, g)-invariant are presented. The constructive proof provides an algorithm for the construction of the feedback law.

Keywords: Controlled invariant distributions; nonlinear control; geometric methods.

1. Introduction

Consider an affine nonlinear system

$$\dot{x}=f(x)+\sum_{i=1}^{m} g_i(x)u_i=f(x)+g(x)u \tag{1.1}$$

where $x \in R^n$, $f(x)$, $g_i(x)$, $i=1,\dots,m$, are C^∞ vector fields. A distribution Δ is said to be (f, g)-invariant, if there exist $\alpha \in C_m^\infty(R^n)$ (m dimensional C^∞ functions) and $\beta \in \mathrm{Gl}(m, C^\infty(R^n))$, such that

$$[\tilde{f}, \Delta] \subset \Delta, \tag{1.2a}$$

$$[\tilde{g}_i, \Delta] \subset \Delta, \quad i=1,\dots,m, \tag{1.2b}$$

where

$$\tilde{f}=f+g\alpha, \qquad \tilde{g}=(\tilde{g}_1,\dots,\tilde{g}_m)=(g_1,\dots,g_m)\beta=g\beta.$$

If for a point $p \in R^n$, there exist a neighborhood U of p, $\alpha \in C_m^\infty(U)$ and $\beta \in \mathrm{Gl}(m, C^\infty(U))$, such that (1.2) holds on U, then Δ is said to be (f, g)-invariant at p.

The concept of (f, g)-invariant distributions plays an important role in the decoupling problems of nonlinear systems and has been investigated widely (see e.g. [5]). This paper deals with local (f, g)-invariance only. As for global aspects, we refer to [1,3].

Since the relations in (1.2) are not verifiable in the general case, it has been the aim of many investigators searching for verifiable equivalent conditions. Let $\mathrm{Span}\{X_1,\dots,X_p\}$ denote the span of vector fields $X_1,\dots,X_p$ over C^∞ functions while $\mathrm{span}\{X_1,\dots,X_p\}$ denotes the span over R. The main result of local (f, g)-invariance is contained in the following theorem [4,6,7,9].

* This research was supported in part by the National Science Foundation under grants ECS-8515899, DMC-8615963 and INT-8519654.

0167-6911/89/$3.50 © 1989, Elsevier Science Publishers B.V. (North-Holland)

Theorem 1.1. *Given a nonsingular and involutive distribution Δ around $p \in R^n$, assume $(\mathcal{G}+\Delta)/\Delta$ is nonsingular at p, where $\mathcal{G}=\mathrm{Span}\{g_1,\ldots,g_m\}$. Then Δ is (f, g)-invariant at p, if and only if*

$$[f, \Delta] \subset \Delta + \mathcal{G}, \tag{1.3a}$$

$$[g_i, \Delta] \subset \Delta + \mathcal{G}, \quad i=1,2,\ldots,m, \tag{1.3b}$$

hold on a neighborhood U of p.

A distribution satisfying (1.3) is called *weakly (f, g)-invariant.*

The concept of (f, g)-invariance is closely related to decoupling problems. For instance, let us consider the disturbance decoupling problem (DDP). Observe the following system:

$$\dot{x}=f(x)+g(x)u+d(x)w, \tag{1.4a}$$

$$y=h(x), \tag{1.4b}$$

where w is an unknown disturbance. The local DDP is finding a feedback control law

$$u=\alpha(x)+\beta(x)v$$

and local coordinates z, such that under coordinates z the feedback system has the following form:

$$\dot{z}^1=f^1(z)+g^1(z)v+d(z)w, \tag{1.5a}$$

$$\dot{z}^2=f^2(z^2)+g^2(z^2)v, \tag{1.5b}$$

$$y=h(z^2). \tag{1.5c}$$

From (1.5) it is clear that the disturbance w does not affect the output y.

The following result follows immediately from Theorem 1.1.

Corollary 1.2. *Let Δ be the largest weakly (f, g)-invariant distribution contained in* $\ker h_*$. *Assume that Δ and $(\mathcal{G}+\Delta)/\Delta$ are nonsingular. Then* DDP *is locally solvable if and only if* $\mathrm{Span}\{d\} \subset \Delta$.

Observe that if we demand a decomposed form like (1.5), a nonsingular and involutive distribution Δ which satisfies $\mathrm{Span}\{d\} \subset \Delta \subset \ker h_*$ is required. However the nonsingularity requirement of $(\mathcal{G}+\Delta)/\Delta$ seems not very reasonable. The following example shows that even if $(\mathcal{G}+\Delta)/\Delta$ is singular, DDP may still be solvable.

Example 1.3. Consider the system

$$\dot{x}_1=f_1(x)+g_1(x)u+d(x)w, \tag{1.6a}$$

$$\dot{x}_2=x_2x_1+x_2\,\mathrm{e}^{x_1}u, \tag{1.6b}$$

$$y=x_2. \tag{1.6c}$$

It is obvious that if we choose $\alpha=-x_1\,\mathrm{e}^{-x_1}$, $\beta=\mathrm{e}^{-x_1}$, DDP is solved. But for system (1.6), $(\mathcal{G}+\Delta)/\Delta$ is singular, where $\Delta=\mathrm{Span}\{\partial/\partial x_1\}$ is the largest weakly (f, g)-invariant distribution contained in $\ker h_*$.

Motivated by the above example we may hope that the condition of nonsingularity of $(\mathcal{G}+\Delta)/\Delta$ can be eliminated from Theorem 1.1. Unfortunately, the next example shows that without the assumption of nonsingularity of $(\mathcal{G}+\Delta)/\Delta$, the conclusion of Theorem 1.1 may not be true.

Example 1.4. Consider the system

$$\dot{x}=\begin{bmatrix}0\\ x_1x_2\end{bmatrix}+\begin{bmatrix}1\\ x_2^3\end{bmatrix}u.$$

Let $\Delta = \mathrm{Span}\{\partial/\partial x_1\}$. Simple computation shows that Δ is weakly (f, g)-invariant. But at $0 \in R^2$, $(\mathcal{G} + \Delta)/\Delta$ is not regular. We claim that Theorem 1.1 does not hold. To see this, we assume that there exists an α, such that $[f + g\alpha, \Delta] \subset \Delta$. It follows that

$$\left(\frac{\partial \alpha}{\partial x_1},\ x_2 + \frac{\partial \alpha}{\partial x_1} x_2^3 \right)^{\mathrm{T}} \in \Delta .$$

That is, for $x_2 \neq 0$, $\alpha = -x_1 x_2^{-1} + \phi(x_2)$. This shows that there does not exist a continuous function α defined on a neighborhood U of 0.

Thus, it is interesting to find a checkable necessary and sufficient condition for a nonsingular and involutive distribution Δ to be (f, g)-invariant without the assumption that $(\mathcal{G} + \Delta)/\Delta$ is nonsingular. This is the main purpose of this paper.

2. Main result

According to Frobenius' Theorem, if a distribution Δ is nonsingular and involutive, then there exists a *local* coordinate chart, such that

$$\Delta = \mathrm{Span}\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right\}. \tag{2.1}$$

Since we consider only the local case, we may restrict our discussion in a Frobenius Cube, i.e. a cube $U = \{x \mid |x_i| < \varepsilon,\ i = 1, \ldots, n\}$ and (2.1) holds in this cube.

If a vector field X is expressed in x coordinates as $X = \sum_{i=1}^n a_i(x)(\partial/\partial x_i)$, we may define

$$X/\Delta = \sum_{i=k+1}^{n} a_i(x) \frac{\partial}{\partial x_i} \tag{2.2}$$

as the natural projection of X on $T(U)/\Delta$.

If a distribution $D = \mathrm{Span}\{Y_1, \ldots, Y_s\}$, then for each $p \in U$ we can define

$$(D/\Delta)_p = \mathrm{Span}\{(Y_1/\Delta)_p, \ldots, (Y_s/\Delta)_p\}$$

which is the natural projection of D on $T(U)/\Delta$. If D/Δ is nonsingular, it is easy to see that

$$D/\Delta = \mathrm{Span}\{Y_1/\Delta, \ldots, Y_s/\Delta\}.$$

Remark 2.1. A coordinate chart (U, x) in which (2.1) holds is a foliation chart in the foliation $\mathcal{F}$ generated by Δ [2]. In fact X/Δ is the projection of X on the quotient manifold M/Δ. Thus it is independent of the choice of the foliation chart (See e.g. pp. 5–10 and p. 28 of [10] and [11]).

Remark 2.2. Let $a(x), b(x) \in C^\infty(R^n)$, $X, Y \in V(R^n)$; then

$$[a(x)X, b(x)Y] = a(x)b(x)[X, Y] + a(x)L_X(b(x))Y - b(x)L_Y(a(x))X. \tag{2.3}$$

Thus, in a Frobenius Cube U,

$$[f, \Delta] \subset \Delta \quad \text{and} \quad [g_j, \Delta] \subset \Delta, \quad j = 1, \ldots, m,$$

are equivalent to

$$\frac{\partial}{\partial x_i}\bar{f} = 0,\ 1 = 1, \ldots, k \quad \text{and} \quad \frac{\partial}{\partial x_i}\bar{g}_j = 0, \quad i = 1, \ldots, k;\ j = 1, \ldots, m,$$

respectively, where $\bar{f} = f/\Delta$ and $\bar{g}_j = g_j/\Delta$.

Next, we present the main result.

Theorem 2.3. *Let Δ be a nonsingular and involutive distribution around $p \in R^n$. Then Δ is (f, g)-invariant at p, if and only if there exists a neighborhood U of p such that on U,*

$$[f, \Delta]/\Delta \subset \operatorname{Span}\{g_j/\Delta \mid j=1,\ldots,m\}, \tag{2.4a}$$

$$[g_i, \Delta]/\Delta \subset \operatorname{Span}\{g_j/\Delta \mid j=1,\ldots,m\}, \quad i=1,\ldots,m, \tag{2.4b}$$

where Span *means the module over $C^\infty(U)$.*

Proof. (*Necessity*) Assume there exists β such that

$$[(g\beta)_s, \Delta] \subset \Delta, \quad s=1,\ldots,m.$$

Then for a basis $\{X_1,\ldots,X_k\}$ of Δ,

$$\left[\sum_{j=1}^{m} g_j\beta_{js}, X_i\right] \triangleq Z_s \in \Delta.$$

It follows that

$$([g_1, X_i],\ldots,[g_m, X_i]) = (g_1,\ldots,g_m)(L_{X_i}\beta)\beta^{-1} + (Z_1,\ldots,Z_m)\beta^{-1}.$$

Thus, we have

$$\begin{aligned}([g_1, X_i]/\Delta,\ldots,[g_m, X_i]/\Delta) &= (g_1/\Delta, g_2/\Delta,\ldots,g_m/\Delta)(L_{X_i}\beta)\beta^{-1}\\ &\in \operatorname{Span}\{(g_1/\Delta), (g_2/\Delta),\ldots,(g_m/\Delta)\}, \quad i=1,2,\ldots,k.\end{aligned}$$

Using the identity (2.3), it follows that for any $X \in \Delta$,

$$[g_i, X]/\Delta \in \operatorname{Span}\{g_j/\Delta \mid j=1,\ldots,m\}, \quad i=1,\ldots,m.$$

Similarly, we can prove (2.4b).

(*Sufficiency*) We prove it by constructing a feedback pair $\{\alpha(x), \beta(x)\}$ in a Frobenius Cube. Assume in the Frobenius Cube $U=\{x \mid |x_i| < \varepsilon, i=1,\ldots,n\}$, that $\Delta = \operatorname{Span}\{\partial/\partial x_1,\ldots,\partial/\partial x_k\}$ and $p=0 \in U$.

First we construct β:

Denote $\bar{g}_i = g_i/\Delta$. Then (2.4b) is equivalent to

$$\frac{\partial}{\partial x_i}(\bar{g}_1, \bar{g}_2,\ldots,\bar{g}_m) = (\bar{g}_1, \bar{g}_2,\ldots,\bar{g}_m)\Gamma_i, \quad i=1,\ldots,k, \tag{2.5}$$

where Γ_i is an $m \times m$ matrix with C^∞ entries.

Following (2.5), we consider the following differential equation:

$$\frac{\partial}{\partial x_i} Y = \Gamma_i^{\mathrm{T}} Y. \tag{2.6}$$

Treating all x_j but x_i as parameters, (2.6) becomes an ordinary differential equation. Thus there exist m linearly independent solutions $Y_1^i,\ldots,Y_m^i$ [8].

Set $W_i = (Y_1^i, Y_2^i,\ldots,Y_m^i)$. Since the rows of $(\bar{g}_1, \bar{g}_2,\ldots,\bar{g}_m)$ are also solutions of (2.6), then from [8] we know that there exists an $m \times (n-k)$ C^∞ matrix L_i such that

$$(\bar{g}_1, \bar{g}_2,\ldots,\bar{g}_m)^{\mathrm{T}} = W_i L_i, \quad i=1,\ldots,k, \tag{2.7}$$

where L_i is independent of x_i which is the independent variable of differential equation (2.6).

Using (2.7), we obtain

$$W_1 L_1 = W_2 L_2 = \cdots = W_k L_k. \tag{2.8}$$

Since W_2 is nonsingular, we have

$$L_2 = W_2^{-1} W_1 L_1. \tag{2.9}$$

Set $x_2 = 0$ on both sides of (2.9). Since L_2 is independent of x_2, we have that $L_2 = W_2^{-1} W_1 |_{x_2=0} \cdot L_1(0, x_3, \ldots, x_n)$. Let $H_2 = W_2^{-1} W_1 |_{x_2=0}$. Then H_2 is a smooth nonsingular matrix. Moreover $L_2 = H_2 L_1(0, x_3, \ldots, x_n)$.

Now, assume H_i is a well defined smooth nonsingular matrix, and

$$L_i(x) = H_i \cdot L_1(1, \ldots, 0, x_{i+1}, \ldots, x_n).$$

Using (2.8), we have

$$L_{i+1}(x) = W_{i+1}^{-1} W_i H_i L_1(0, \ldots, 0, x_{i+1}, \ldots, x_n).$$

Recall that $L_{i+1}(x)$ is independent of x_{i+1} and let

$$H_{i+1} = W_{i+1}^{-1} W_i H_i |_{x_{i+1}} = 0;$$

then we have

$$L_{i+1}(x) = H_{i+1} \cdot L_1(0, \ldots, 0, x_{i+2}, \ldots, x_n).$$

Thus, we may construct $\{H_i\}$ recursively. Finally we have H_k and

$$L_k(x) = H_k L_1(0, \ldots, 0, x_{k+1}, \ldots, x_n).$$

Let $H = W_k H_k$, $L = L_1(0, \ldots, 0, x_{k+1}, \ldots, x_n)$. Then H is a smooth nonsingular matrix, and L is independent of $x_1, \ldots, x_k$. Moreover $(\bar{g}_1, \ldots, \bar{g}_m)^{\mathrm{T}} = HL$.

Now we define

$$\beta = (H^{\mathrm{T}})^{-1} = (H_k^{\mathrm{T}} W_k^{\mathrm{T}})^{-1}. \tag{2.10}$$

Then

$$(\bar{g}_1, \ldots, \bar{g}_m)\beta = L^{\mathrm{T}}$$

is independent of $x_1, \ldots, x_k$. Thus we have

$$\frac{\partial}{\partial x_i}(\bar{g}\beta)_j = 0. \tag{2.11}$$

That is $[(g\beta)_j, \Delta] \subset \Delta$, $j = 1, \ldots, m$.

Next we prove (1.2a):

Note that (2.4a) is equivalent to

$$\frac{\partial}{\partial x_i}\bar{f} = (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_m)\xi_i, \quad i = 1, \ldots, k, \tag{2.12}$$

where $\bar{f} = f/\Delta$, ξ_i is an $m \times 1$ vector with C^∞ entries.

It follows from (2.10)–(2.12) that

$$\bar{g}\beta \frac{\partial H^{\mathrm{T}}\xi_j}{\partial x_i} = \bar{g}\beta \frac{\partial H^{\mathrm{T}}\xi_i}{\partial x_j}. \tag{2.13}$$

We will construct α as follows:

$$\alpha \triangleq -(H^{\mathrm{T}})^{-1}\Big[\int_0^{x_k}(H^{\mathrm{T}}\xi_k)(x_1, \ldots, x_{k-1}, \tau)\,\mathrm{d}\tau + \int_0^{x_{k-1}}(H^{\mathrm{T}}\xi_{k-1})\cdot(x_1, \ldots, x_{k-2}, \tau, 0)\,\mathrm{d}\tau + \cdots + \int_0^{x_1}(H^{\mathrm{T}}\xi_1)(\tau, 0, \ldots, 0)\,\mathrm{d}\tau\Big]. \tag{2.14}$$

Using (2.3), we may show that

$$\frac{\partial}{\partial x_i}(\bar{f}+\bar{g}\alpha) = \bar{g}\xi_i - \bar{g}\xi_i = 0. \qquad \square$$

Note that the form of the projection X/Δ etc. depends on the chosen coordinate frame. If the coordinate frame at hand is a flat chart, i.e., a foliation chart in $\mathscr{F}$, then the verification of (2.4) can be performed easily. Otherwise it may be difficult to verify (2.4). Thus to avoid using the above projection we would like to find an equivalent condition of (2.4).

Assume we have a local basis of Δ, say

$$\Delta = \mathrm{Span}\{X_1, \ldots, X_k\}. \tag{2.15}$$

By solving algebraic equations, we may obtain a dual basis of $\Omega = \Delta^\perp$ as

$$\Delta^\perp = \Omega = \mathrm{Span}\{w_1, \ldots, w_{n-k}\}. \tag{2.16}$$

Now for any vector field Y we denote

$$\langle w, Y\rangle = \begin{bmatrix} \langle w_1, Y\rangle \\ \vdots \\ \langle w_{n-k}, Y\rangle \end{bmatrix}$$

which is a column vector of $n-k$ C^∞ functions. Then we have:

Corollary 2.4. *Let Δ be the same distribution as in Theorem 2.3. Then Δ is (f, g)-invariant at p, if and only if on a neighborhood U of p,*

$$\langle w, [f, X_j]\rangle \in \mathrm{Span}\{\langle w, g_1\rangle, \ldots, \langle w, g_m\rangle\}, \quad j = 1, \ldots, k, \tag{2.17a}$$

$$\langle w, [g_i, X_j]\rangle \in \mathrm{Span}\{\langle w, g_1\rangle, \ldots, \langle w, g_m\rangle\}, \quad i = 1, \ldots, m;\ j = 1, \ldots, k. \tag{2.17b}$$

Proof. Choose a flat chart (U, ϕ) sich that

$$\Delta = \mathrm{Span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right\} \quad \text{and} \quad \Omega = \Delta^\perp = \mathrm{Span}\{\mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_n\}.$$

If follows from the definition of the projection X/Δ etc. that (2.4) is equivalent to (2.17) with these special bases.

Since $\langle w_i, X_j\rangle = 0$, it is easy to see that (2.17) is independent of the choice of the basis $\{X_1, \ldots, X_k\}$ of Δ. By the linearity of the inner product over C^∞ functions one sees that (2.17) is independent of the choice of the basis $\{w_1, \ldots, w_{n-k}\}$ of Ω too. Finally, since (2.17) is independent of the local coordinates, the conclusion follows. $\square$

3. Algorithms for computing $\alpha(X)$ and $\beta(x)$ in Frobenius' Cube

The constructive proof of Theorem 2.3 supplies an algorithm for constructing feedback pair $\{\alpha(x), \beta(x)\}$ in a Frobenius Cube. We summarize it as follows.

Algorithm 3.1.

1. Verify $\partial\bar{g}/\partial x_i = \bar{g}\Gamma_i$, $\partial\bar{f}/\partial x_i = \bar{g}\xi_i$, $i = 1, \ldots, k$, by finding Γ_i and ξ_i.
2. Find W_i by solving

$$\frac{\partial}{\partial x_i} Y = \Gamma_i^{\mathrm{T}} Y, \quad Y(0) = I_m, \tag{3.1}$$

and let $W_i = (Y_1^i, \ldots, Y_m^i)$. Note that $Y(0)$ can be any other nonsingular matrix.

3. Define $H_1, \dots, H_k$ recursively as

$$H_1 = I, \qquad H_{s+1} = W_{s+1}^{-1} W_s H_s \,|_{x_{s+1}=0}, \quad s \geq 1.$$

Let $H = W_k H_k$.

4. Construct α and β as

$$\alpha = \text{R.H.S. of (2.14)}, \qquad \beta = \left(H_k^{\mathrm{T}} W_k^{\mathrm{T}}\right)^{-1} = \left(H^{\mathrm{T}}\right)^{-1}.$$

Following example shows the working procedures of the above algorithm.

Example 3.2. Consider the system

$$\dot{x} = \begin{bmatrix} f^1(x) \\ f^2(x) \\ x_2 x_3 \\ x_4 \sin x_1 \end{bmatrix} + \begin{bmatrix} g_1^1(x) \\ g_2^1(x) \\ x_3 \cos x_1 \\ -x_4 \mathrm{e}^{x_2} \sin x_1 \end{bmatrix} u_1 + \begin{bmatrix} g_1^2(x) \\ g_2^2(x) \\ x_3 \sin x_1 \\ x_4 \mathrm{e}^{x_2} \cos x_1 \end{bmatrix} u_2.$$

$$\Delta = \operatorname{Span}\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\}.$$

It is clear that $(\mathscr{G} + \Delta)/\Delta$ is singular at $0 \in R^4$.

It is easy to verify that

$$\frac{\partial}{\partial x_1}[\bar{g}_1 \quad \bar{g}_2] = \begin{bmatrix} -x_3 \sin x_1 & x_3 \cos x_1 \\ -x_4 \mathrm{e}^{x_2} \cos x_1 & -x_4 \mathrm{e}^{x_2} \sin x_1 \end{bmatrix} = [\bar{g}_1 \quad \bar{g}_2] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \triangleq \bar{g}\Gamma_1.$$

Similarly,

$$\Gamma_2 = \begin{bmatrix} \sin^2 x_1 & -\sin x_1 \cos x_1 \\ -\sin x_1 \cos x_1 & \cos^2 x_1 \end{bmatrix}, \qquad \xi_1 = \begin{bmatrix} -\sin x_1 \cos x_1 \, \mathrm{e}^{-x_2} \\ \cos^2 x_1 \, \mathrm{e}^{-x_2} \end{bmatrix}, \qquad \xi_2 = \begin{bmatrix} \cos x_1 \\ \sin x_1 \end{bmatrix}.$$

Now, solving

$$\frac{\mathrm{d}W_1}{\mathrm{d}x_1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\mathrm{T}} W_1 \triangleq \Gamma_1^{\mathrm{T}} W_1,$$

we have

$$W_1 = \begin{bmatrix} \cos x_1 & -\sin x_1 \\ \sin x_1 & \cos x_1 \end{bmatrix}.$$

Likewise, we have

$$W_2 = \begin{bmatrix} \cos x_1 & \sin x_1 \\ \sin x_1 & -\cos x_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{e}^{x_2} \end{bmatrix} \begin{bmatrix} \cos x_1 & \sin x_1 \\ \sin x_1 & -\cos x_1 \end{bmatrix}.$$

Then

$$H_2 = W_2^{-1} W_1 H_1 \,|_{x_2=0} = W_1.$$

$$\beta = \left(H_2^{\mathrm{T}} W_2^{\mathrm{T}}\right)^{-1} = \begin{bmatrix} \cos x_1 & -\mathrm{e}^{-x_2} \sin x_1 \\ \sin x_1 & \mathrm{e}^{-x_2} \cos x_1 \end{bmatrix}, \qquad \alpha = \begin{bmatrix} -x_2 \cos x_1 + \mathrm{e}^{-x_2} \sin^2 x_1 \\ -x_2 \sin x_1 - \mathrm{e}^{-x_2} \sin x_1 \cos x_1 \end{bmatrix}.$$

Remark 3.3. In fact the feedback law $\{\alpha(x), \beta(x)\}$ is not unique. The previous method gives a pair of $\alpha(x)$ and $\beta(x)$ only. Using this algorithm one may obtain different $\alpha(x)$ and $\beta(x)$ because the Γ_i and ξ_i are not unique when $\dim((\mathscr{G}+\Delta)/\Delta) < m$.

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