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Simultaneous Stabilization for a Collection of Multi-input Nonlinear Systems *

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Abstract This paper considers the problem of designing a controller that simultaneously stabilizes a collection of multi-input nonlinear systems. Based on the technique of control Lyapunov function, a sufficient condition for the existence of time-invariant simultaneously stabilizing state feedback controller is obtained. A universal formula for constructing a continuous controller which simultaneously stabilizes the systems is derived. The result extends the existing one in Ref. [1] for the single-input case.

Key words simultaneous stabilization, affine nonlinear system, control Lyapunov function, state feedback

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1 Introduction

The simultaneous stabilization problem is important in the field of robust control. The problem is designing a single controller which simultaneously stabilizes a finite collection of systems. The simultaneous stabilization problem was firstly introduced in Ref. [2] and Ref. [3]. It was showed that this problem reduces to a strong stabilization problem in the case of two plants. For linear systems, some necessary and sufficient conditions for the existence of simultaneously stabilizing state feedback and output feedback controllers have been obtained, see e.g., Refs. [4～6], and references therein. For nonlinear systems, the simultaneous stabilization problem is more difficult. Ref. [7] and Ref. [1] have presented some results for nonlinear systems. Ref. [7] designed a continuous state feedback controller which simultaneously stabilizes a countable family of nonlinear systems, and provided a sufficient condition for the existence of simultaneously asymptotically stabilizing controller for a collection of nonlinear systems. However, the designed controller is not easy to implement for it is constructed as a sum of infinite time-varying functions. Moreover, the sufficient condition is difficult to verify because it requires firstly finding an asymptotically stabilizing state feedback controller for each system, then deriving a corresponding Lyapunov function for each closed-loop system, and finally determining whether some infinite sequences of time-varying functions exist that satisfy some specified conditions. Ref. [1] designed a more applicable controller for simultaneously stabilizing a collection of single-input affine nonlinear systems, and provided necessary and sufficient conditions for the existence of

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simultaneously stabilizing state feedback controller. Similar to Ref. [1], applying the control Lyapunov function (see e.g., Ref. [8] and Ref. [9]), this paper extends the results in Ref. [1] to multi-input affine nonlinear systems.

The paper is organized as follows. Section 2 gives some preliminaries. Section 3 presents sufficient conditions for the existence of simultaneously globally asymptotically stabilizing state feedback controller for a collection of multi-input affine nonlinear systems, and provides a universal formula for simultaneously stabilizing controller. Section 4 gives an illustrative example. Section 5 is the conclusion.

2 Preliminaries

Consider an affine nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, f(0) = 0, \tag{1}$$

with the state $x \in \mathbf{R}^n$ and the control input $u = (u_1, \cdots, u_m) \in \mathbf{R}^m$, where vector fields f and g_i, $i = 1, \cdots, m$, are smooth (i.e. f, g_i's $\in C^r$ for some suitable $r > 0$).

Definition 2.1[8] A smooth, proper, and positive definite function V is a control Lyapunov function (CLF) for system (1), if for all $x \in \mathbf{R}^n \setminus \{0\}$

$$\inf_{u \in \mathbf{R}^m} \left\{ \nabla V(x) \cdot \left[f(x) + \sum_{i=1}^{m} g_i(x) u_i \right] \right\} < 0. \tag{2}$$

Denote $a(x) \doteq \nabla V(x) \cdot f(x), b_i(x) \doteq \nabla V(x) \cdot g_i(x), i = 1, \cdots, m,$

and set $B(x) \doteq (b_1(x), \cdots, b_m(x)), \beta(x) \doteq \| B(x) \|^2 = \sum_{i=1}^{m} b_i^2(x).$

Then (2) is equivalent to the following expression in Ref. [8] $\beta(x) = 0 \Rightarrow a(x) < 0.$

Definition 2.2[8] A CLF V for system (1) is said to be satisfying small control property, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $\| x \| < \delta$, then there is some u with $\| u \| < \varepsilon$ such that

$$\nabla V(x) \cdot \left[f(x) + \sum_{i=1}^{m} g_i(x) u_i \right] < 0. \tag{3}$$

Denote

$$\phi(a(x), \beta(x)) \doteq \begin{cases} \dfrac{a(x) + \sqrt{a^2(x) + \beta^2(x)}}{\beta(x)}, & \text{if } \beta(x) \neq 0, \\ 0, & \text{if } \beta(x) = 0, \end{cases} \tag{4}$$

$$p_i(x) \doteq \begin{cases} -b_i(x) \phi(a(x), \beta(x)), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{5}$$

Theorem 2.3[8] $p_i(x)$ is continuous if the CLF V for the system (1) satisfies the small control property, and the control law $p = (p_1, \cdots, p_m)$ globally asymptotically stabilizes system (1).

3 Simultaneous stabilization

Consider a collection of affine nonlinear systems

$$\Sigma_i : \dot{x} = f_i(x) + \sum_{j=1}^{m} g_{ij}(x) u_j, f_i(0) = 0, i = 1, 2, \cdots, q, \tag{6}$$

where the state $x \in \mathbf{R}^n$, the control input $u = (u_1, \cdots, u_m) \in \mathbf{R}^m$, and for each $i \in \{1, 2, \cdots, q\}$, vector fields $f_i(x)$ and $g_i(x) = (g_{i1}(x), \cdots, g_{im}(x))$ are smooth.

Assume $V_i(x)$ is a CLF for the system Σ_i, $i = 1, \cdots, q$. For $i = 1, \cdots, q$ and $j = i, \cdots m$, and set

$$a_i(x) \doteq \nabla V_i(x) \cdot f_i(x), b_{ij}(x) \doteq \nabla V_i(x) \cdot g_{ij}(x), B_i(x) \doteq (b_{i1}(x), \cdots, b_{im}(x)),$$

$$\beta_i(x) \doteq \|B_i(x)\|^2 = \sum_{j=1}^{m} b_{ij}^2(x).$$

For $j = 1, \cdots, m$, define

$$I_j^P(x) \doteq \{i \in \{1, \cdots, q\} \mid b_{ij}(x) > 0\}, I_j^N(x) \doteq \{i \in \{1, \cdots, q\} \mid b_{ij}(x) < 0\},$$
$$I_j^Z(x) \doteq \{i \in \{1, \cdots, q\} \mid b_{ij}(x) = 0\}.$$

and

$$D_j^Z \doteq \{x \in \mathbf{R}^n \mid I_j^P(x) = \varnothing \text{ and } I_j^N(x) = \varnothing\}, D_j^P \doteq \{x \in \mathbf{R}^n \mid I_j^P(x) \neq \varnothing \text{ and } I_j^N(x) = \varnothing\},$$
$$D_j^N \doteq \{x \in \mathbf{R}^n \mid I_j^P(x) \neq \varnothing \text{ and } I_j^N(x) \neq \varnothing\}, D_j^M \doteq \{x \in \mathbf{R}^n \mid I_j^P(x) \neq \varnothing \text{ and } I_j^N(x) \neq \varnothing\}.$$

where $\varnothing$ denotes the empty set. Obviously,

$$I_j^P(x) \cup I_j^N(x) \cup I_j^Z(x) = \{1, \cdots, q\}, \text{for each } j \in \{1, \cdots, m\}, \tag{7}$$

and

$$D_j^Z \cup D_j^P \cup D_j^N \cup D_j^M = \mathbf{R}^n, \text{for each } j \in \{1, \cdots, m\}. \tag{8}$$

Moreover, for each $j \in \{1, \cdots, m\}$, sets I_j^P, I_j^N, I_j^Z are disjoint, and sets D_j^Z, D_j^P, D_j^N, D_j^M are also disjoint. Function $V_i(x)$, $i = 1, \cdots, q$, are smooth, so the origin $x = 0$ belongs to D_j^Z, $j = 1, \cdots, m$.

For $i = 1, \cdots, q$, and $j = 1, \cdots, m$, take

$$\phi_i(a_i(x), \beta_i(x)) \doteq \begin{cases} \dfrac{a_i(x) + \sqrt{a_i^2(x) + \beta_i^4(x)}}{\beta_i(x)}, & \text{if } \beta_i(x) \neq 0, \\ 0, & \text{if } \beta_i(x) = 0. \end{cases} \tag{9}$$

$$p_{ij}(x) \doteq \begin{cases} -b_{ij}(x)\phi_i(a_i(x), \beta_i(x)), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{10}$$

For each $j \in \{1, \cdots, m\}$, let

$$u_j^P(x) \doteq \begin{cases} \min\limits_{i \in I_j^P(x)} \left\{ -b_{ij}(x) \dfrac{a_i(x)}{\beta_i(x)} \right\}, & \text{if } I_j^P(x) \neq \varnothing, \\ +\infty, & \text{if } I_j^P(x) = \varnothing. \end{cases}$$

$$u_j^N(x) \doteq \begin{cases} \max\limits_{i \in I_j^N(x)} \left\{ -b_{ij}(x) \dfrac{a_i(x)}{\beta_i(x)} \right\}, & \text{if } I_j^N(x) \neq \varnothing, \\ -\infty, & \text{if } I_j^N(x) = \varnothing. \end{cases}$$

and

$$u_j^M(x) \doteq \begin{cases} \dfrac{1}{2}(u_j^P(x) + u_j^N(x)), & \text{if } x \in D_j^M, \\ \text{undefined}, & \text{elsewhere}. \end{cases} \tag{11}$$

For each $i \in \{1, \cdots, q\}$, denote

$$\beta_i^Z \doteq \{x \in \mathbf{R}^n \mid \beta_i(x) = 0\}.$$

The following result gives a sufficient condition for the existence of simultaneously globally asymptotically stabilizing state feedback controller for a collection of multi-input affine nonlinear systems. It extends the corresponding result in Ref. [1].

Theorem 3.1 Consider the collection of affine nonlinear systems (6). If there exists a collection of CLFs $V_i(x)$, $i = 1, \cdots, q$, satisfying the small control property, such that for all $j \in \{1, \cdots, m\}$, and for all $x \in D_j^M$, the following conditions are satisfied.

(1) $$\max_{i \in I_j^N(x)} \left\{ -b_{ij}(x) \frac{a_i(x)}{\beta_i(x)} \right\} < \min_{i \in I_j^P(x)} \left\{ -b_{ij}(x) \frac{a_i(x)}{\beta_i(x)} \right\}. \tag{12}$$

(2) If $i \in I_j^Z(x)$, then $x \in \beta_i^Z$ (i.e., $i \in \bigcap_{j=1}^m I_j^Z(x)$).

(3) For all $i \in \{1, \cdots, q\}$

$$\lim_{x\to\bar{x}} \frac{\beta_i(x)}{b_{ij}(x)} = 0, \tag{13}$$

where $\bar{x} \in \beta_i^Z$.

Then there exists a continuous state feedback control law $p(x) = (p_1(x), \cdots, p_m(x))$, with

$$p_j(x) = \begin{cases} p_j^Z(x), & \text{if} \quad x \in D_j^Z, \\ p_j^P(x), & \text{if} \quad x \in D_j^P, \\ p_j^N(x), & \text{if} \quad x \in D_j^N, \\ p_j^M(x), & \text{if} \quad x \in D_j^M, \end{cases} \tag{14}$$

which simultaneously globally asymptotically stabilizes the collection of affine nonlinear systems (6), where

$$p_j^Z(x) = 0,$$

$$p_j^P(x) = \begin{cases} \min\limits_{i\in I_j^P(x)} p_{ij}(x), & \text{if} \quad I_j^P(x) \neq \varnothing, \\ \text{undefined}, & \text{if} \quad I_j^P(x) = \varnothing, \end{cases}$$

$$p_j^N(x) = \begin{cases} \max\limits_{i\in I_j^N(x)} p_{ij}(x), & \text{if} \quad I_j^N(x) \neq \varnothing, \\ \text{undefined}, & \text{if} \quad I_j^N(x) = \varnothing, \end{cases}$$

and

$$p_j^M(x) = \begin{cases} \min(p_j^N(x), \max(p_j^P(x), u_j^M(x))), & \text{if} \ x \in D_j^M, \\ \text{undefined}, & \text{elsewhere}. \end{cases}$$

Proof We prove this theorem in two steps. First, we show the feedback control law $u = p(x)$ simultaneously stabilizes the collection of affine nonlinear systems (6). That is, to show the following inequalities

$$a_i(x) + \sum_{j=1}^{m} b_{ij}(x)\, p_j(x) < 0, \quad \forall x \in \mathbf{R}^n \setminus \{0\}, \forall i \in \{1, \cdots, q\} \tag{15}$$

hold. Second, we show the continuity of the vector function $p(x)$.

A. Simultaneous stabilization

Ⅰ. If $x \in D_j^Z \setminus \{0\}$, for each $j \in \{1, \cdots, m\}$, then $p_j(x) = p_j^Z(x) = 0$, $b_{ij}(x) = 0$, $\forall i \in \{1, \cdots, q\}$.

So $b_{ij}(x)\, p_j(x) = 0$, $\forall i \in \{1, \cdots, q\}$.

Ⅱ. If $x \in D_j^P$, for each $j \in \{1, \cdots, m\}$, then $p_j(x) = p_j^P(x) = \min\limits_{i\in I_j^P(x)} p_{ij}(x)$.

(1) If $i \in I_j^P(x)$, then $b_{ij}(x) > 0$, so $b_{ij}(x)\, p_j(x) < b_{ij}(x)\, p_{ij}(x) < -b_{ij}^2(x)\dfrac{a_i(x)}{\beta_i(x)}$.

(2) If $i \in I_j^Z(x)$, then $b_{ij}(x) = 0$, so $b_{ij}(x)\, p_j(x) = 0$.

Ⅲ. If $x \in D_j^N$, for each $j \in \{1, \cdots, m\}$, then $p_j(x) = p_j^N(x) = \max\limits_{i\in I_j^N(x)} p_{ij}(x)$.

(1) If $i \in I_j^N(x)$, then $b_{ij}(x) < 0$, so $b_{ij}(x)\, p_j(x) < b_{ij}(x)\, p_{ij}(x) < -b_{ij}^2(x)\dfrac{a_i(x)}{\beta_i(x)}$.

(2) If $i \in I_j^Z(x)$, then $b_{ij}(x) = 0$, so $b_{ij}(x)\, p_j(x) = 0$.

Ⅳ. If $x \in D_j^M$, for each $j \in \{1, \cdots, m\}$, then $p_j(x) = p_j^M(x)$.

Note that for all $x \in D_j^M$, the inequality (12) holds, that is

$$-b_{ij}(x)\frac{a_i(x)}{\beta_i(x)} < -b_{kj}(x)\frac{a_k(x)}{\beta_k(x)}, \ \forall i \in I_j^N(x) \text{ and } \forall k \in I_j^P(x).$$

According to the definition of $p_j^M(x)$, obviously, the inequality

$$\max_{i\in I_j^N(x)}\left\{-b_{ij}(x)\frac{a_i(x)}{\beta_i(x)}\right\} < p_j^M(x) < \min_{i\in I_j^P(x)}\left\{-b_{ij}(x)\frac{a_i(x)}{\beta_i(x)}\right\}$$

holds. Then

(1) $b_{ij}(x)p_j(x) < -b_{ij}^2(x)\dfrac{a_i(x)}{\beta_i(x)}, \quad \forall i \in I_j^P(x)$.

(2) $b_{ij}(x)p_j(x) < -b_{ij}^2(x)\dfrac{a_i(x)}{\beta_i(x)}, \quad \forall i \in I_j^N(x)$.

Furthermore, note that $b_{ij}(x)p_j(x) = 0, \quad \forall i \in I_j^Z(x)$. Hence, for each $j \in \{1, \cdots, m\}$, we have

(1) If $x \in D_j^Z \setminus \{0\}$, then $b_{ij}(x)p_j(x) = 0, \forall i \in \{1, \cdots, q\}$.

(2) If $x \in D_j^P \cup D_j^N \cup D_j^M$, then

① $b_{ij}(x)p_j(x) < -b_{ij}^2(x)\dfrac{a_i(x)}{\beta_i(x)}, \quad \forall i \in \{1, \cdots, q\} \setminus I_j^Z(x)$,

② $b_{ij}(x)p_j(x) = 0$, for all $i \in I_j^Z(x)$.

Therefore, for any given $x \in \mathbf{R}^n \setminus \{0\}$, according to (8), we have

(1) If $x \in \bigcap_{j=1}^m D_j^Z$, then

$$a_i(x) + \sum_{j=1}^m b_{ij}(x)p_j(x) = a_i(x) < 0, \forall i \in \{1, \cdots, q\}.$$

(2) If there exists at least a $j \in \{1, \cdots, m\}$ such that $x \notin D_j^Z$, then $x \in D_j^P \cup D_j^N \cup D_j^M$. Denote

$$J \coloneqq \{j \in \{1, \cdots, m\} \mid x \in D_j^P \cup D_j^N \cup D_j^M\}.$$

Additionally, note that $\forall j \in \{1, \cdots, m\} \setminus J$, $b_{ij}(x) = 0$, $b_{ij}(x)p_j(x) = 0$, $\forall i \in \{1, \cdots, q\}$. Therefore

① $a_i(x) + \sum_{j=1}^m b_{ij}(x)p_j(x) < a_i(x) + \sum_{j=1}^m -b_{ij}^2(x)\dfrac{a_i(x)}{\beta_i(x)} = 0, \forall i \in \{1, \cdots, q\} \setminus \bigcap_{j\in J} I_j^Z(x)$.

② $a_i(x) + \sum_{i=1}^m b_{ij}(x)p_j(x) = a_i(x) < 0, \forall i \in \bigcap_{j\in J} I_j^Z(x)$.

Summarizing the above, we see that for any give $x \in \mathbf{R}^n \setminus \{0\}$

$$a_i(x) + \sum_{j=1}^m b_{ij}(x)p_j(x) < 0, \forall i \in \{1, \cdots, q\}.$$

Therefore, the feedback control law $u = p(x)$ simultaneously globally asymptotically stabilizes the collection of affine nonlinear systems (6).

B. Continuity of vector function $p(x)$

To prove the continuity of vector function $p(x)$ it is enough to show $p_j(x)$, $j = 1, \cdots, m$, the components of $p(x)$, are continuous. The proof of the continuity of $p_j(x)$, $j = 1, \cdots, m$, is similar to that in Ref. [1].

According to Theorem 2.3, for each $i \in \{1, \cdots, q\}$, the function $p_{ij}(x)$, $j = 1, \cdots, m$, are continuous if the CLF $V_i(x)$ for the affine nonlinear system (6) satisfies the small control property, so with (8) and (14), it is enough to prove the continuity of $p_j(x)$, $j = 1, \cdots, m$, on the boundary between the set $\{x \in \mathbf{R}^n \mid b_{ij}(x) = 0\}$ and the set $\{x \in \mathbf{R}^n \mid b_{ij}(x) \neq 0\}$, for each $i \in \{1, \cdots, q\}$. For convenience, we prove $P_j(x)$, $j = 1, \cdots, m$, are continuous in the interior of D_j^Z, D_j^P, D_j^N, D_j^M, and on the boundaries between them.

First, we show $p_j(x)$, $j = 1, \cdots, m$, are continuous in the interior of D_j^Z, D_j^P, D_j^N, D_j^M.

For $j = 1, \cdots, m$, obviously, $p_j^Z(x) = 0$ is continuous in the interior of D_j^Z. Since for each $i \in \{1, \cdots, q\}$, the function $p_{ij}(x)$ is continuous, $p_j^P(x)$ is continuous in the interior of D_j^P, and $p_j^N(x)$ is continuous in the interior of D_j^N. In the following, we prove $p_j^M(x)$ is continuous in the interior of D_j^M. Let $D_{ij}^{MP} \coloneqq \{x \in D_j^M \mid b_{ij}(x) > 0\}$, for each $i \in I_j^P(x)$. For any $\bar{x} \in D_j^M$ on the boundary of D_{ij}^{MP} (i.e., $b_{ij}(\bar{x}) = 0$), take a sequence of vector $\{x_k\} \in D_{ij}^{MP}$, such that $x_k \to \bar{x}$, as $k \to +\infty$. Since $\bar{x} \in D_j^M$, so $I_j^P(\bar{x}) \neq \varnothing$, then there exists an $l \neq i$ such that $b_{lj}(x_k) > 0$ for sufficient large k. Since $i \in I_j^P(\bar{x})$, according to the Condition 2, $\bar{x} \in \beta_i^Z$ (i.e., $i \in \bigcap_{j=1}^m I_j^Z(\bar{x})$), then $a_i(x_k)$

<0 for sufficient large k, and with (13) $-b_{ij}(x_k)\dfrac{a_i(x_k)}{\beta_i(x_k)} \to +\infty$ as $x_k \to \bar{x}$. So

$$\lim_{k\to+\infty} u_j^P(x_k) = \lim_{k\to+\infty} \min_{i\in I_j^P(x_k)} \left\{ -b_{ij}(x_k)\frac{a_i(x_k)}{\beta_i(x_k)} \right\} = \lim_{k\to+\infty} \min_{i\in I_j^P(x_k)\setminus \cap_{j=1}^m I_j^Z(\bar{x})} \left\{ -b_{ij}(x_k)\frac{a_i(x_k)}{\beta_i(x_k)} \right\}$$

$$= \min_{i\in I_j^P(\bar{x})} \left\{ -b_{ij}(\bar{x})\frac{a_i(\bar{x})}{\beta_i(\bar{x})} \right\} = u_j^P(\bar{x})$$

This implies $u_j^P(x)$ is continuous on the boundary, contained in D_j^M, of D_{ij}^{MP}. Similarly, $u_j^N(x)$ is continuous on the boundary, contained in D_j^M, of $D_{ij}^{MN} \doteq \{x \in D_j^M \mid b_{ij}(x) < 0\}$, for each $i \in I_j^N(x)$. Hence $u_j^M(x)$ is continuous in the interior of D_j^M, and $P_j^M(x)$ is continuous in the interior of D_j^M.

Next, we prove $p_j(x)$, $j=1,\cdots,m$, are continuous on the boundaries between $D_j^Z, D_j^P, D_j^N, D_j^M$.

(1) For any $\bar{x} \in D_j^Z$ on the boundary between D_j^Z and D_j^P (i.e., $b_{ij}(\bar{x}) = 0$ for all $i \in \{1,\cdots,q\}$), take a sequence of vector $\{x_k\} \in D_j^P$, such that $x_k \to \bar{x}$, as $k \to +\infty$. Then

$$\lim_{k\to+\infty} p_j^P(x_k) = \lim_{k\to+\infty} \min_{i\in I_j^P(x_k)} p_{ij}(x_k) = 0 = p_j^Z(\bar{x}). \tag{16}$$

This implies $p_j(x)$ is continuous on the boundary between D_j^P and D_j^Z.

(2) Similarly, $p_j(x)$ is continuous on the boundary between D_j^N and D_j^Z.

(3) For any $\bar{x} \in D_j^Z$ on the boundary between D_j^Z and D_j^M (i.e., $b_{ij}(\bar{x}) = 0$, $\forall i \in \{1,\cdots,q\}$), take a sequence of vector $\{x_k\} \in D_j^M$, such that $x_k \to \bar{x}$, as $k \to +\infty$. Then

$$\lim_{k\to+\infty} p_j^M(x_k) = \lim_{k\to+\infty} \min(p_j^N(x_k), \max(p_j^P(x_k), u_j^M(x_k)))$$

$$= \min(0, \max(0, \lim_{k\to+\infty} u_j^M(x_k))) = 0 = p_j^Z(\bar{x}). \tag{17}$$

This implies $p_j(x)$ is continuous on the boundary between D_j^M and D_j^Z.

(4) For any $\bar{x} \in D_j^P$ on the boundary between D_j^P and D_j^M (i.e., $b_{ij}(\bar{x}) \geq 0$, $\forall i \in \{1,\cdots,q\}$), take a sequence of vector $\{x_k\} \in D_j^M$, such that $x_k \to \bar{x}$, as $k \to +\infty$. Since $\bar{x} \in D_j^P$, $I_j^P(\bar{x}) \neq 0$ and $I_j^N(\bar{x}) = \varnothing$. Then with (13), we have

$$u_j^P = \min_{i\in I_j^P(x_k)} \left\{ -b_{ij}(x_k)\frac{a_i(x_k)}{\beta_i(x_k)} \right\} \text{ is finite as } k \to +\infty, \tag{18}$$

and

$$u_j^N(x_k) = \max_{i\in I_j^N(x_k)} \left\{ -b_{ij}(x_k)\frac{a_i(x_k)}{\beta_i(x_k)} \right\} \to -\infty \text{ as } k \to +\infty.$$

So $u_j^M(x_k) \to -\infty$ as $k \to +\infty$. Therefore,

$$\lim_{k\to+\infty} p_j^M(x_k) = \lim_{k\to+\infty} \min(p_j^N(x_k), \max(p_j^P(x_k), u_j^M(x_k)))$$

$$= \min(0, \max(\lim_{k\to+\infty} p_j^P(x_k), -\infty)) = \lim_{k\to+\infty} p_j^P(x_k) = p_j^P(\bar{x}).$$

This implies $p_j(x)$ is continuous on the boundary between D_j^M and D_j^P.

(5) For any $\bar{x} \in D_j^N$ on the boundary of D_j^N (i.e., $b_{ij}(\bar{x}) \leq 0$, $\forall i \in \{1,\cdots,q\}$), take a sequence of vector $\{x_k\} \in D_j^M$, such that $x_k \to \bar{x}$, as $k \to +\infty$. Since $\bar{x} \in D_j^N$, $I_j^P(\bar{x}) = \varnothing$ and $I_j^N(\bar{x}) \neq \varnothing$. Then with (13), we have

$$u_j^N(x_k) = \max_{i\in I_j^N(x_k)} \left\{ -b_{ij}(x_k)\frac{a_i(x_k)}{\beta_i(x_k)} \right\} \text{ is finite as } k \to +\infty, \tag{19}$$

and

$$u_j^P(x_k) = \min_{i\in I_j^N(x_k)} \left\{ -b_{ij}(x_k)\frac{a_i(x_k)}{\beta_i(x_k)} \right\} \to +\infty \text{ as } k \to +\infty.$$

So $u_j^M(x_k) \to +\infty$ as $k \to +\infty$. Therefore,

$$\lim_{k\to+\infty} p_j^M(x_k) = \lim_{k\to+\infty}\min(p_j^N(x_k), \max(p_j^P(x_k), u_j^M(x_k)))$$
$$= \min(\lim_{k\to+\infty} p_j^N(x_k), \max(0, +\infty)) = \lim_{k\to+\infty} p_j^N(x_k) = p_j^N(\bar{x}).$$

This implies $p_j(x)$ is continuous on the boundary between D_j^M and D_j^N.

(6) According to the definition, the boundary between D_j^P and D_j^N belongs to D_j^Z. It has been proved that $p_j(x)$ is continuous in both $D_j^P \cup D_j^Z$ and $D_j^N \cup D_j^Z$, so it must be continuous on the boundary between D_j^P and D_j^N.

(7) According to equation (16) and equation (17), for any $\bar{x} \in D_j^Z$ belongs to the boundary between D_j^P, D_j^Z and D_j^M, $p_j(x)$ is continuous at $\bar{x}$. This implies $p_j(x)$ is continuous on the boundary between D_j^P, D_j^Z and D_j^M. Similarly, $p_j(x)$ is continuous on the boundary between D_j^N, D_j^Z and D_j^M, and continuous on the boundary between D_j^P, D_j^Z and D_j^N. Note that boundary between D_j^P, D_j^N and D_j^M belongs to $D_Z{}^j$. Since $p_j(x)$ is continuous in both $D_j^P \cup D_j^Z \cup D_j^M$ and $D_j^N \cup D_j^Z \cup D_j^M$, it must be continuous on the boundary between D_j^P, D_j^N and D_j^M. Therefore, it is also continuous on the boundary between D_j^P, D_j^N, D_j^Z and D_j^M.

Thus, the proof is completed.

Remark 3.2 Assume the Conditions (2) and (3) of Theorem 3.1 are respectively replaced by

(2)′
$$I_j^Z(x) = \varnothing. \tag{20}$$

(3)′ For all $i \in \{1, \cdots, q\}$
$$\left|\lim_{x\to\bar{x}} \frac{\beta_i(x)}{b_{ij}(x)}\right| < +\infty, \tag{21}$$

where $\bar{x} \in \beta_i^Z$.

Then the feedback control law $p(x)$ can also simultaneously globally asymptotically stabilize the collection of the systems (6). In fact, if the Condition (2)′ holds, then p_j^M is obviously continuous in the interior of D_j^M, and if the Condition (3)′ holds, then equation (18) and equation (19) are still valid.

4 An illustrative example

Consider the following three affine nonlinear systems:

$$S_1: \begin{cases} \dot{x}_1 = x_1 x_2, \\ \dot{x}_2 = -x_2 - u_1 - u_2. \end{cases}$$
$$S_2: \begin{cases} \dot{x}_1 = -x_1 - 2x_1x_2(x_1^2 + x_2^2) + x_1u_1 + x_1u_2, \\ \dot{x}_2 = x_2u_1 + x_2u_2. \end{cases}$$
$$S_3: \begin{cases} \dot{x}_1 = -x_1x_2, \\ \dot{x}_2 = -x_2 - u_1 - u_2. \end{cases}$$

Let $x = [x_1 \ x_2]^T$, and take
$$V_1(x) = \frac{1}{2}[x_1^2 + (x_2 + x_1^2)^2],$$
$$V_2(x) = \frac{1}{2}(x_1^2 + x_2^2),$$
$$V_3(x) = \frac{1}{2}[x_1^2 + (x_2 - x_1^2)^2].$$

Then
$$\dot{V}_1(x)|_{S_1} = -x_2^2 + 2x_1^2x_2(x_2 + x_1^2) - (x_2 + x_1^2)u_1 - (x_2 + x_1^2)u_2,$$
$$\dot{V}_2(x)|_{S_2} = -x_1^2 - 2x_1^2x_2(x_1^2 + x_2^2) + (x_1^2 + x_2^2)u_1 + (x_1^2 + x_2^2)u_2,$$
$$\dot{V}_3(x)|_{S_3} = -x_2^2 + 2x_1^2x_2(x_2 - x_1^2) - (x_2 - x_1^2)u_1 - (x_2 - x_1^2)u_2.$$

Take $u_1 = u_2 = x_1^2 x_2$, then

$$\dot{V}_1(x)|_{S_1} < 0, \dot{V}_2(x)|_{S_2} < 0, \text{ and } \dot{V}_3(x)|_{S_3} < 0, \forall x \in \mathbf{R}^n \setminus \{0\},$$

so $V_1(x)$, $V_2(x)$, and $V_3(x)$ are CLFs for the S_1, S_2 and S_3 respectively. Let

$$a_1(x) = -x_2^2 + 2x_1^2 x_2(x_2 + x_1^2),$$
$$a_2(x) = -x_1^2 - 2x_1^2 x_2(x_1^2 + x_2^2),$$
$$a_3(x) = -x_2^2 + 2x_1^2 x_2(x_2 - x_1^2),$$

and for $j = 1, 2$

$$b_{1j}(x) = -(x_2 + x_1^2),$$
$$b_{2j}(x) = x_1^2 + x_2^2,$$
$$b_{3j}(x) = -(x_2 - x_1^2).$$

Take $\beta_i(x) = b_{i1}^2(x) + b_{i2}^2(x)$, $i = 1, 2, 3$. Since $\lim_{\|x\|\to 0}\sup \frac{a_i(x)}{\sqrt{\beta_i(x)}} = 0$, $i = 1, 2, 3$,

$V_1(x)$, $V_2(x)$ and $V_3(x)$ satisfy the small control property. Then for $j = 1, 2$,

$$D_j^Z = \{x \in \mathbf{R}^2 \mid x_1 = 0, \text{and } x_2 = 0\},$$
$$D_j^P = \{x \in \mathbf{R}^2 \mid x_2 \leq -x_1^2, \text{and } x \neq 0\},$$
$$D_j^N = \varnothing,$$
$$D_j^M = \{x \in \mathbf{R}^2 \mid x_2 > -x_1^2\}.$$

Denote

$$D_j^{M_1} = \{x \in \mathbf{R}^2 \mid x_2 > x_1^2\},$$
$$D_j^{M_2} = \{x \in \mathbf{R}^2 \mid -x_1^2 < x_2 < x_1^2\},$$
$$D_j^{M_3} = \{x \in \mathbf{R}^2 \mid x_2 = x_1^2\},$$

then $D_j^{M_1} \cup D_j^{M_2} \cup D_j^{M_3} = D_j^M$ $(j = 1, 2)$. Moreover,

(1) if $x \in D_j^{M_1}$, then $I_j^P(x) = \{2\}$, and $I_j^N(x) = \{1, 3\}$;

(2) if $x \in D_j^{M_2}$, then $I_j^P(x) = \{2, 3\}$, and $I_j^N(x) = \{1\}$;

(3) if $x \in D_j^{M_3}$, then $I_j^P(x) = \{2\}$, $I_j^N(x) = \{1\}$, and $I_j^Z(x) = \{3\}$.

For $j = 1, 2$, if $x \in D_j^{M_1} \cup D_j^{M_2} \cup D_j^{M_3}$, then

$$-b_{2j}(x)\frac{a_2(x)}{\beta_2(x)} - \left[-b_{1j}(x)\frac{a_1(x)}{\beta_1(x)}\right] = \frac{1}{2}\left(\frac{x_1^2}{x_1^2 + x_2^2} + \frac{x_2^2}{x_2 + x_1^2}\right) > 0.$$

If $x \in D_j^{M_1}$, then

$$-b_{2j}(x)\frac{a_2(x)}{\beta_2(x)} - \left[-b_{3j}(x)\frac{a_3(x)}{\beta_3(x)}\right] = \frac{1}{2}\left(\frac{x_1^2}{x_1^2 + x_2^2} + \frac{x_2^2}{x_2 - x_1^2}\right) > 0.$$

If $x \in D_j^{M_2}$, then

$$-b_{3j}(x)\frac{a_3(x)}{\beta_3(x)} - \left[-b_{1j}(x)\frac{a_1(x)}{\beta_1(x)}\right] = \frac{1}{2}\left(-\frac{2x_1^2 x_2^2}{x_2 - x_1^2}\right) > 0.$$

Summarizing the above, for $j = 1, 2$, and all $x \in D_j^M$, Condition (1) of Theorem 3.1, i.e., the inequality (12) holds. Obviously, for $x \in D_j^{M_3}$, we have $3 \in I_j^Z(x)$, $x \in \beta_3^Z$, so the Condition (2) of Theorem 3.2 holds. Additionally, It is easy to see that the Condition (3) of Theorem 3.1 also holds. Let

$$\phi_1(a_1(x), \beta_1(x)) := \begin{cases} \dfrac{a_1(x) + \sqrt{a_1^2(x) + \beta_1^4(x)}}{\beta_1(x)}, & \text{if } x_2 \neq -x_1^2, \\ 0, & \text{if } x_2 = -x_1^2, \end{cases}$$

$$\phi_2(a_2(x), \beta_2(x)) := \begin{cases} \dfrac{a_2(x) + \sqrt{a_2^2(x) + \beta_2^4(x)}}{\beta_2(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

$$\phi_3(a_3(x), \beta_3(x)) := \begin{cases} \dfrac{a_3(x) + \sqrt{a_3^2(x) + \beta_3^4(x)}}{\beta_3(x)}, & \text{if } x_2 \neq x_1^2, \\ 0, & \text{if } x_2 = x_1^2, \end{cases}$$

and for $i = 1,2,3$, $j = 1,2$, let
$$p_{ij}(x) = \begin{cases} -b_{ij}(x)\,\Phi(a_i(x), \beta_i(x)), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

For $j = 1,2$, set $p_j^Z(x) = 0$, if $x = 0$,

$$p_j^P(x) = \begin{cases} \min(p_{1j}(x), p_{2j}(x), p_{3j}(x)), & \text{if } x_2 < -x_1^2, \\ \min(p_{2j}(x), p_{3j}(x)), & \text{if } -x_1^2 \le x_2 < x_1^2, \\ p_{2j}(x), & \text{if } x_2 \ge x_1^2, \text{and } x \neq 0, \\ \text{undefined}, & \text{if } x = 0, \end{cases}$$

$$p_j^N(x) = \begin{cases} \max(p_{1j}(x), p_{3j}(x)), & \text{if } x_2 > x_1^2, \\ p_{1j}(x), & \text{if } -x_1^2 < x_2 \le x_1^2, \text{and } x \neq 0, \\ \text{undefined}, & \text{if } x_2 \le -x_1^2, \end{cases}$$

$$p_j^M(x) = \begin{cases} \min(p_j^N, (x), \max(p_j^P(x), u_j^M(x))), & \text{if } x_2 > -x_1^2, \\ \text{undefined}, & \text{if } x_2 \le -x_1^2, \end{cases}$$

where

$$u_j^M(x) = \begin{cases} \frac{1}{2}\left(-b_{2j}\frac{a_2(x)}{\beta_2(x)} + \max\left\{-b_{1j}\frac{a_1(x)}{\beta_1(x)}, -b_{3j}\frac{a_3(x)}{\beta_3(x)}\right\}\right), & \text{if } x_2 > x_1^2, \\ \frac{1}{2}\left(\min\left\{-b_{2j}\frac{a_2(x)}{\beta_2(x)}, -b_{3j}\frac{a_3(x)}{\beta_3(x)}\right\} - b_{1j}\frac{a_1(x)}{\beta_1(x)}\right), & \text{if } -x_1^2 < x_2 < x_1^2, \\ \frac{1}{2}\left(-b_{1j}\frac{a_1(x)}{\beta_1(x)} - b_{2j}\frac{a_2(x)}{\beta_2(x)}\right), & \text{if } x_2 = x_1^2, \text{and } x \neq 0, \\ \text{undefined}, & \text{if } x_2 \le -x_1^2. \end{cases}$$

Therefore, according to Theorem 3.1, there exists a continuous feedback control law $p(x) = (p_1(x), p_2(x))$, with
$$p_j(x) = \begin{cases} p_j^Z(x), & \text{if } x = 0, \\ p_j^P(x), & \text{if } x_2 \le -x_1^2, \text{and } x \neq 0, \\ p_j^M(x), & \text{if } x_2 > -x_1^2, \end{cases}$$
simultaneously globally asymptotically stabilizes the systems S_1, S_2, and S_3.

Remark 3.3 If we just consider simultaneous stabilization for systems S_1 and S_2, it is easy to see that sets D_j^Z, D_j^P, D_j^N are the same as the above, and for all $x \in D_j^M$, Conditions (1), (2)′, and (3)′ are satisfied. So according to Remark 3.2, there exists a continuous feedback control law, which simultaneously globally asymptotically stabilizes the systems S_1, and S_2.

In fact, for $j = 1,2$, let

$$p_j^{P'}(x) = \begin{cases} \min(p_{1j}(x), p_{2j}(x)), & \text{if } x_2 < -x_1^2, \\ p_{2j}(x), & \text{if } x_2 \ge -x_1^2, \text{and } x \neq 0, \\ \text{undefined}, & x = 0, \end{cases}$$

$$p_j^{N'}(x) = \begin{cases} p_{1j}(x), & \text{if } x_2 > -x_1^2, \\ \text{undefined}, & \text{if } x_2 \le -x_1^2, \end{cases}$$

$$P_j^{M'}(x) = \begin{cases} \min(p_j^{N'}(x), \max(p_j^{P'}(x), u_j^{M'}(x))), & \text{if } x_2 > -x_1^2, \\ \text{undefined}, & \text{if } x_2 \le -x_1^2, \end{cases}$$

where

$$u_j^{M'}(x) = \begin{cases} \frac{1}{2}\left(-b_{1j}(x)\frac{a_1(x)}{\beta_1(x)} - b_{2j}(x)\frac{a_2(x)}{\beta_2(x)}\right), & \text{if } x_2 > -x_1^2, \\ \text{undefined}, & \text{if } x_2 \le -x_1^2. \end{cases}$$

According to Remark 3.2, there exists a continuous feedback control law $p'(x) = (p_1'(x), p_2'(x))$, with

$$p_j'(x) = \begin{cases} p_j^{Z}(x), & \text{if } x = 0, \\ p_j^{P'}(x), & \text{if } x_2 \leq -x_1^2, \text{and } x \neq 0, \\ p_j^{M'}(x), & \text{if } x_2 > x_1^2, \end{cases}$$

simultaneously globally asymptotically stabilizes the systems S_1 and S_2.

5 Conclusion

The paper considered the problem of simultaneous stabilization for a collection of multi-input affine nonlinear systems. Sufficient conditions for the existence of the controller have been obtained, and a feedback controller has been designed. An illustrative example was included to demonstrate the design procedure.

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一组多输入非线性系统的同时镇定

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摘　要　考虑设计一个控制器同时镇定一组多输入非线性系统的问题. 利用控制李雅普诺夫函数方法,给出了时不变同时镇定状态反馈控制器存在的充分条件,然后给出构造同时镇定一组系统的连续控制器的统一公式. 该结果是对文献[1]中讨论的单输入系统的结论的推广.

关键词　同时镇定,仿射非线性系统,控制李雅普诺夫函数,状态反馈