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On Hamiltonian realization of time-varying nonlinear systems

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This paper investigates Hamiltonian realization of time-varying nonlinear (TVN) systems, and proposes a number of new methods for the problem. It is shown that every smooth TVN system can be expressed as a generalized Hamiltonian system if the origin is the equilibrium of the system. If the Jacobian matrix of a TVN system is nonsingular, the system has a generalized Hamiltonian realization whose structural matrix and Hamiltonian function are given explicitly. For the case that the Jacobian matrix is singular, this paper provides a constructive decomposition method, and then proves that a TVN system has a generalized Hamiltonian realization if its Jacobian matrix has a nonsingular main diagonal block. Furthermore, some sufficient (necessary and sufficient) conditions for dissipative Hamiltonian realization of TVN systems are also presented in this paper.

generalized Hamiltonian realization, dissipative Hamiltonian realization, diffoemorphism, structural construction

1 Introduction

Energy-based control and stability analysis have been extensively studied for a wide range of physical systems, which include robotic manipulators[1], flexible link robots[2], surface vehicles[3], space crafts[4], mechanical systems[5], electrical systems[6], etc. While much research work in the area of robotics leads to a good understanding of this approach, its recent successful applications to power systems require new problem formulations and new insights into this approach. In recent years, Port-Controlled Hamiltonian (PCH) systems[7,8] have been well investigated in a series of works[9−12]. The Hamiltonian function, the sum of potential energy (excluding gravitational potential energy) and kinetic energy in physical systems, is a good candidate of Lyapunov functions for many physical systems, and has been successfully applied to the control of power systems in some recent works[13−16].

In order to apply the energy-based approach, it is important to be able to express the system under

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consideration as a Hamiltonian system, i.e., to obtain the Generalized Hamiltonian Realization (GHR) for the system[17]. GHR is a very difficult issue for general nonlinear systems, since it involves solving a kind of partial differential equation which is usually very difficult to solve[18]. For time-invariant nonlinear systems, the GHR problem has been studied recently in a series of works[11,17−20]. In ref. [11], some elegant and useful results were obtained for feedback GHR, after thorough investigation of interconnection and damping assignment passivity-based control of PCH systems. Using the normal form, an effective approximation approach was provided for the GHR of time-invariant systems in ref. [17]. Orthogonal decomposition and control switching methods were investigated in ref. [18]. Indeed, there are many GHR results for time-invariant systems. However, for TVN system, the Hamiltonian realization problem remains to be investigated.

In this paper, we study the Hamiltonian realization problem for TVN systems. Using coordinate transformation and structural construction techniques, several new methods are developed in this paper. The main results of this paper are as follows:

(i) It is shown that every smooth TVN system can be expressed as a generalized Hamiltonian system if the origin is the equilibrium of the system.

(ii) If the Jacobian matrix of a TVN system is nonsingular, the system has a GHR whose both structural matrix and Hamiltonian function are given explicitly.

(iii) A constructive decomposition method is proposed for the case that the Jacobian matrix of a TVN system is singular. Using the method, we show that a TVN system has a GHR, if its Jacobian matrix has a nonsingular main diagonal block.

(iv) Some sufficient (necessary and sufficient) conditions are provided for dissipative Hamiltonian realization of TVN systems.

The rest of the paper is organized as follows. Section 2 gives some concepts and properties. Section 3 deals with the GHR of TVN systems, and provides some sufficient (necessary and sufficient) conditions for the problem. In section 4, we present several new results for the dissipative Hamiltonian realization, which is followed by the conclusion in section 5.

2 Concepts and properties

To facilitate the analysis, some fundamental concepts and properties are listed in this section for the system under study.

Definition 1. Let $\mathcal{M}$ denote an n-dimensional manifold. Then, a time-varying nonlinear system described by

$$\dot{x} = f(x,t), \quad x \in \mathcal{M}, \quad t \in \mathbb{R}^+ := [0, +\infty) \tag{1}$$

is said to have a generalized Hamiltonian realization (GHR), if there exists a suitable structural matrix $T(x,t) \in \mathbb{R}^{n\times n}$ and a Hamiltonian function $H(x,t)$ such that system (1) can be expressed as

$$\dot{x} = T(x,t)\frac{\partial H(x,t)}{\partial x}. \tag{2}$$

We decompose the structure matrix as

$$T(x,t) = J(x,t) + P(x,t), \tag{3}$$

where $J(x,t) \in \mathbb{R}^{n\times n}$ is skew-symmetric and $P(x,t) \in \mathbb{R}^{n\times n}$ is symmetric. Furthermore, assume that x is a regular point of $P(x,t)$ in the sense that there exists a neighborhood, Ω, of x such that the number of positive eigenvalues and the number of negative eigenvalues are invariant for $x \in \Omega$ and $t \in \mathbb{R}^+$. Then, we may further decompose $P(x,t)$ at regular point x as

$$P(x,t) = -R(x,t) + S(x,t), \tag{4}$$

where $0 \leqslant R(x,t) \in \mathbb{R}^{n\times n}$, $0 \leqslant S(x,t) \in \mathbb{R}^{n\times n}$ and the ranks of $R(x,t)$ and $S(x,t)$ are equal to the numbers of positive eigenvalues and negative eigenvalues of $P(x,t)$, respectively. Thus, at the regular point of $P(x,t)$, we have the following unique decomposition:

$$T(x,t) = J(x,t) - R(x,t) + S(x,t). \tag{5}$$

Definition 2. System (1) is said to have a dissipative Hamiltonian realization if it can be expressed as (2) with $T(x,t) = J(x,t) - R(x,t)$, i.e., $S(x,t) \equiv 0$ in the structural matrix decomposition (5).

Proposition 1. System (2) is a dissipative Hamiltonian realization, if and only if $T(x,t)+T(x,t)^T \leqslant 0$.

Proof. It follows immediately from Definition 2.

Proposition 2. Assume that $T(x,t)$ in (2) can be decomposed as $T(x,t) = J(x,t) - R(x,t)$, with skew-symmetric $J(x,t)$ and $R(x,t) \geqslant 0$. If $\frac{\partial H}{\partial t} \leqslant 0$ and there exists a $\mathcal{K}$-function α such that $H(x,t) \geqslant \alpha(\|x\|) > 0, \forall x \neq 0$, then system (2) is Lyapunov stable.

Proof. Consider Hamiltonian function $H(x,t) \geqslant \alpha(\|x\|) > 0$ as a Lyapunov function candidate. Its derivative is given by

$$\dot{H} = \frac{\partial H^T}{\partial x}\Big[J(x,t) - R(x,t)\Big]\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t} = -\frac{\partial H^T}{\partial x}R(x,t)\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t} \leqslant 0.$$

Thus, system (2) is Lyapunov stable.

3 Generalized Hamiltonian realization

In this section, we will present several new results for the generalized Hamiltonian realization of system (1). First, we propose some useful technical terms which are essential for our further development.

Assume that $f(x),\ x \in \mathbb{R}^n$, is a scalar function. As it is well known, the first-order partial derivative of $f(x)$ can be defined as gradient $\nabla f(x) := \partial f/\partial x$ and the second partial derivative can be given by the Hessian matrix $\mathrm{Hess}(f(x))$. A natural extension is the definition of partial derivatives of arbitrarily any order, say, n.

In the following, we develop a method which gives arbitrary order partial derivative of a function or function matrix easily. Some nice properties of the method are also presented.

Definition 3. Let

$$\frac{\partial}{\partial x} = \Big[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\Big]^T, \ \ x \in \mathbb{R}^n.$$

Then, the unified partial derivative operator (UPDO) is defined recursively as

$$\frac{\partial^m}{\partial x^m} = \frac{\partial^{m-1}}{\partial x^{m-1}} \otimes \frac{\partial}{\partial x}, \quad m \geqslant 1, \tag{6}$$

where $\otimes$ is the Kronecker product, and the products between elements are defined as

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}} \cdot \frac{\partial}{\partial x_j} := \frac{\partial^{i_1+\cdots+(i_j+1)+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_j^{i_j+1}\cdots\partial x_n^{i_n}},$$

$$i_1+\cdots+i_n = m-1, \ \ j = 1,2,...,n,$$

in which, $\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}$ $(i_1 + ... + i_n = m-1)$ and $\frac{\partial}{\partial x_j}$ are arbitrary components of $\frac{\partial^{m-1}}{\partial x^{m-1}}$ and $\frac{\partial}{\partial x}$, respectively. For completeness, we further define $\frac{\partial^0}{\partial x^0} := \mathcal{I}$, which is called the identity operator and satisfies

$$\mathcal{I} \otimes \frac{\partial^s}{\partial x^s} = \frac{\partial^s}{\partial x^s} \otimes \mathcal{I} = \frac{\partial^s}{\partial x^s}, \ \ \forall\, s \geqslant 1.$$

Remark 1. From the definition above, it is easy to know that $\frac{\partial^m}{\partial x^m}$, $m \geqslant 1$, is an n^m-dimensional column vector operator. For example,

$$\frac{\partial^2}{\partial x^2} = \Big[\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_1 \partial x_2}, \cdots, \frac{\partial^2}{\partial x_1 \partial x_n}, \cdots, \frac{\partial^2}{\partial x_n \partial x_1}, \frac{\partial^2}{\partial x_n \partial x_2}, \cdots, \frac{\partial^2}{\partial x_n^2}\Big]^T$$

is an n^2-dimensional column vector.

Remark 2. It is interesting to note that though the first-order UPDO is the same as the gradient operator ∇, the second-order UPDO, $\frac{\partial^2}{\partial x^2}$, is no longer equal to the operator Hess $(\cdot)$.

With the definition above, given a scalar function $f(x)$,

$$f(x) \otimes \frac{\partial^m}{\partial x^m} := \frac{\partial^m f(x)}{\partial x^m}, \ \ m \geqslant 1 \tag{7}$$

is well defined. Based on (7), given a vector field $X(x) \in \mathbb{R}^p$, we can define

$$\frac{\partial^m X(x)}{\partial x^m} := X(x) \otimes \frac{\partial^m}{\partial x^m}, \ \ m \geqslant 1. \tag{8}$$

From (8), it is easy to show that the following recursive formula holds for the higher order derivative operators

$$\frac{\partial^m X(x)}{\partial x^m} = \frac{\partial}{\partial x}\Big(\frac{\partial^{m-1} X(x)}{\partial x^{m-1}}\Big), \ \ m \geqslant 1, \tag{9}$$

where $\frac{\partial^0 X(x)}{\partial x^0} := X(x) \otimes \mathcal{I} = X(x)$.

In fact, from (6) and (8), we have

$$\begin{aligned} \frac{\partial^m X(x)}{\partial x^m} &= X(x) \otimes \frac{\partial^m}{\partial x^m} = X(x) \otimes \Big(\frac{\partial^{m-1}}{\partial x^{m-1}} \otimes \frac{\partial}{\partial x}\Big) \\ &= \Big(X(x) \otimes \frac{\partial^{m-1}}{\partial x^{m-1}}\Big) \otimes \frac{\partial}{\partial x} = \Big(\frac{\partial^{m-1} X(x)}{\partial x^{m-1}}\Big) \otimes \frac{\partial}{\partial x} \\ &= \frac{\partial}{\partial x}\Big(\frac{\partial^{m-1} X(x)}{\partial x^{m-1}}\Big). \end{aligned}$$

Thus, (9) holds.

Similarly, given a function matrix $A(x) \in \mathbb{R}^{p\times q}$, based on (7), we define

$$\frac{\partial^m A(x)}{\partial x^m} := A(x) \otimes \frac{\partial^m}{\partial x^m}, \ \ \ m \geqslant 1. \tag{10}$$

It can also be shown that

$$\frac{\partial^m A(x)}{\partial x^m} = \frac{\partial}{\partial x}\Big(\frac{\partial^{m-1} A(x)}{\partial x^{m-1}}\Big), \ \ m \geqslant 1, \tag{11}$$

where $\frac{\partial^0 A(x)}{\partial x^0} := A(x) \otimes \mathcal{I} = A(x)$.

Lemma 1. Assume that $A(x) \in \mathbb{R}^{p\times q}$ and $B(x) \in \mathbb{R}^{q\times l}$ are smooth function matrices. Then,

$$\frac{\partial}{\partial x}\Big[A(x)B(x)\Big] = \frac{\partial A(x)}{\partial x}B(x) + \Big[A(x) \otimes I_n\Big]\frac{\partial B(x)}{\partial x}, \tag{12}$$

where I_n is the $n \times n$ identity matrix.

Proof. Let $A(x) = [a_{ij}(x)]$, $B(x) = [b_{ij}(x)]$, and $C(x) := A(x)B(x) = [c_{ij}(x)] \in \mathbb{R}^{p\times l}$. Then,

$$\frac{\partial c_{ij}(x)}{\partial x} = \frac{\partial}{\partial x}\Big[\sum_{s=1}^{q} a_{is}(x)b_{sj}(x)\Big] = \sum_{s=1}^{q}\Big[\frac{\partial a_{is}(x)}{\partial x}b_{sj}(x) + a_{is}(x)\frac{\partial b_{sj}(x)}{\partial x}\Big].$$

Thus, the rth component of $\frac{\partial c_{ij}(x)}{\partial x}$, i.e., the element located at $(n(i-1)+r,\, j)$ in $\frac{\partial C(x)}{\partial x}$ is given as

$$\sum_{s=1}^{q}\Big[\frac{\partial a_{is}(x)}{\partial x_r}b_{sj}(x) + a_{is}(x)\frac{\partial b_{sj}(x)}{\partial x_r}\Big],\ \ i,\ j,\ r = 1, 2, ..., n. \tag{13}$$

On the other hand, the element located at $(n(i-1)+r,\, j)$ in the matrix on the right-hand side of (12) is as follows:

$$\begin{aligned}&\sum_{s=1}^{q}\Big[\frac{\partial a_{is}(x)}{\partial x_r}b_{sj}(x)\Big] + \sum_{s=1}^{q}\Big[\alpha_{is}^{(r)}(x)\frac{\partial b_{sj}(x)}{\partial x}\Big]\\ &= \sum_{s=1}^{q}\Big[\frac{\partial a_{is}(x)}{\partial x_r}b_{sj}(x)\Big] + \sum_{s=1}^{q}\Big[a_{is}(x)\frac{\partial b_{sj}(x)}{\partial x_r}\Big],\ \ i,\ j,\ r = 1, 2, ..., n,\end{aligned} \tag{14}$$

where $\alpha_{is}^{(r)}(x) := \Big[0, \cdots, \underbrace{a_{is}(x)}_{r\text{th}}, \cdots, 0\Big]$ is an n-dimensional row vector with its rth component being $a_{is}(x)$ and others being zero. From (13) and (14), we know that (12) holds. QED

Before we present the high-degree homogeneous factorization lemma for functions (see Lemma 2), let us introduce a few preparing definitions first.

The high-order semi-tensor products of x are defined recursively as

$$x^{[m]} = x^{[m-1]} \otimes x,\ \ m \geqslant 1,\ \ x^{[0]} := 1,\ \ x \in \mathbb{R}^n. \tag{15}$$

Using $x^{[m]}$, we now construct an $n^m \times n^m$ matrix, E_{n^m}, which is obtained from I_{n^m} by certain element re-arrangement ($m \geqslant 1$). The construction steps are given as follows:

Step 1. Establish a map $\psi : \{1, 2, ..., n^m\} \longmapsto x^{[m]}$ by $\psi(k) = k$th component of $x^{[m]}$.

Step 2. Define $\psi_{\min}(k) = \min\{j \mid \psi(j) = \psi(k)\}$.

Step 3. Swap the element '1' at (k, k) in I_{n^m} with '0' at $(k, \psi_{\min}(k))$, if $\psi_{\min}(k) < k$, $k = 1, 2, ..., n^m$.

With the preparation above, we have the following high-degree homogeneous factorization lemma (HHF-Lemma, for short).

Lemma 2 (HHF-Lemma). Assume that $f(x,t)$ ($x \in \mathbb{R}^n$, $t \geqslant 0$) is a scalar function. If $f(x,t)$ has continuous mth-order partial derivatives with respect to x and satisfies $\frac{\partial^{s-1}}{\partial x^{s-1}} f(0,t) = 0$, $s = 1, 2, ..., m$, then there exists a vector $\alpha(x,t) \in \mathbb{R}^{n^m}$ such that

$$f(x,t) = \alpha^T(x,t)\, x^{[m]}. \tag{16}$$

Proof. The proof is completed by mathematical induction on m. First, let us prove that it is true for case $m = 1$.

$$\begin{aligned}&\Big(\int_0^1 \frac{\partial f(\xi_1, \xi_2, ..., \xi_n; t)}{\partial \xi_1}\Big|_{\xi_1 = x_1 s, ...,\ \xi_n = x_n s} ds\Big)\, x_1\\ &= \int_0^1 \frac{\partial f(\xi_1, \xi_2, ...,\ \xi_n; t)}{\partial \xi_1}\Big|_{\xi_1 = x_1 s, ...,\ \xi_n = x_n s} d\xi_1\\ &= \Big(f(\xi_1, ..., \xi_n; t) + \psi(\xi_2, ..., \xi_n; t)\Big)\Big|_0^1 = f(x,t) + \psi(x_2, ..., x_n; t) - \psi(0,t).\end{aligned} \tag{17}$$

Let $f^{(1)}(x_2, ..., x_n; t) \stackrel{\triangle}{=} \psi(x_2, ..., x_n; t) - \psi(0, t)$, then (17) becomes

$$f(x,t) = \left(\int_0^1 \frac{\partial f(\xi_1, \xi_2, ..., \xi_n; t)}{\partial \xi_1} \Big|_{\xi_1 = x_1 s, ..., \ \xi_n = x_n s} ds \right) x_1 - f^{(1)}(x_2, ..., x_n; t). \tag{18}$$

Since $f^{(1)}(0, t) = 0$ holds, similar to the above, we can obtain

$$f^{(1)}(x_2, ..., x_n; t) = \left(\int_0^1 \frac{\partial f^{(1)}(\xi_2, ..., \xi_n; t)}{\partial \xi_2} \Big|_{\xi_2 = x_2 s, ..., \ \xi_n = x_n s} ds \right) x_2 - f^{(2)}(x_3, ..., x_n; t). \tag{19}$$

Substituting (19) into (18) gives

$$f(x,t) = a_1(x,t) x_1 + a_2(x,t) x_2 + f^{(2)}(x_3, ..., x_n; t), \tag{20}$$

where

$$a_1(x,t) := \int_0^1 \frac{\partial f(\xi_1, \xi_2, ..., \xi_n; t)}{\partial \xi_1} \Big|_{\xi_1 = x_1 s, ..., \ \xi_n = x_n s} ds,$$
$$a_2(x,t) := - \int_0^1 \frac{\partial f^{(1)}(\xi_2, ..., \xi_n; t)}{\partial \xi_2} \Big|_{\xi_2 = x_2 s, ..., \ \xi_n = x_n s} ds.$$

Continuing with $f^{(2)}(x_3, ..., x_n; t)$, until (20) becomes

$$\begin{aligned} f(x,t) &= \sum_{i=1}^{n-1} a_i(x,t) x_i + f^{(n-1)}(x_n; t) \\ &= \sum_{i=1}^{n-1} a_i(x,t) x_i + \Big(\int_0^1 \frac{\partial f^{(n-1)}(\xi_n; t)}{\partial \xi_n} \Big|_{\xi_n = x_n s} ds \Big) x_n = \sum_{i=1}^{n} a_i(x,t) x_i, \end{aligned} \tag{21}$$

where $a_n(x,t) = \displaystyle\int_0^1 \frac{\partial f^{(n-1)}(\xi_n; t)}{\partial \xi_n} \Big|_{\xi_n = x_n s} ds$. Eq. (21) means that the lemma holds for case $m = 1$.

Next, let us assume that the lemma holds for case $m-1$, i.e., there exists a vector $\gamma^T(x,t) \in \mathbb{R}^{n^{(m-1)}}$ such that

$$f(x,t) = \gamma(x,t)\ x^{[m-1]}. \tag{22}$$

Then, we only need to show that the lemma holds true for case m.

From the construction of E_{n^m}, it is easy to see that

$$\gamma(x,t)\ x^{[m-1]} = \Big[\gamma(x,t)\ E_{n^{m-1}} \Big] x^{[m-1]}.$$

Let $\beta(x,t) := \gamma(x,t)\, E_{n^{m-1}}$. Then, (22) becomes

$$f(x,t) = \beta(x,t) x^{[m-1]}. \tag{23}$$

From (23) and Lemma 1, we can recursively obtain the following:

$$\begin{aligned} \frac{\partial f(x,t)}{\partial x} &= \frac{\partial \beta(x,t)}{\partial x} x^{[m-1]} + \Big(\beta(x,t) \otimes I_n \Big) \frac{\partial x^{[m-1]}}{\partial x}, \\ \frac{\partial^2 f(x,t)}{\partial x^2} &= \frac{\partial}{\partial x} \Big(\frac{\partial \beta(x,t)}{\partial x} x^{[m-1]} \Big) + \frac{\partial}{\partial x} \Big[\Big(\beta(x,t) \otimes I_n \Big) \frac{\partial x^{[m-1]}}{\partial x} \Big] \\ &= \frac{\partial^2 \beta(x,t)}{\partial x^2} x^{[m-1]} + \Big[\frac{\partial \beta(x,t)}{\partial x} \otimes I_n + \frac{\partial}{\partial x} \Big(\beta(x,t) \otimes I_n \Big) \Big] \frac{\partial x^{[m-1]}}{\partial x} \\ &\quad + \Big(\beta(x,t) \otimes I_{n^2} \Big) \frac{\partial^2 x^{[m-1]}}{\partial x^2}, \end{aligned}$$

$$\vdots$$

$$\begin{aligned}
\frac{\partial^{m-1} f(x,t)}{\partial x^{m-1}} &= \frac{\partial^{m-1}\beta(x,t)}{\partial x^{m-1}} x^{[m-1]} \\
&+ \Big[\frac{\partial^{m-2}\beta(x,t)}{\partial x^{m-2}} \otimes I_n + \frac{\partial}{\partial x}\Big(\frac{\partial^{m-3}\beta(x,t)}{\partial x^{m-3}} \otimes I_n\Big) \\
&+ \cdots + \frac{\partial^{m-2}}{\partial x^{m-2}}\Big(\beta(x,t)\otimes I_n\Big)\Big] \frac{\partial}{\partial x} x^{[m-1]} \\
&+ \cdots \\
&+ \Big[\frac{\partial(\beta \otimes I_{n^{m-4}})}{\partial x} \otimes I_{n^2} + \frac{\partial}{\partial x}\Big(\beta \otimes I_{n^{m-3}}\Big) \otimes I_n \\
&+ \frac{\partial}{\partial x}\Big(\beta \otimes I_{n^{m-2}}\Big)\Big] \frac{\partial^{m-2}}{\partial x^{m-2}} x^{[m-1]} \\
&+ \Big(\beta(x,t) \otimes I_{n^{m-1}}\Big) \frac{\partial^{m-1}}{\partial x^{m-1}} x^{[m-1]}.
\end{aligned} \tag{24}$$

Since $\frac{\partial^{m-1}}{\partial x^{m-1}} f(0,t) = 0$ and $\left(\frac{\partial^i}{\partial x^i} x^{[m-1]}\right)\Big|_{x=0} = 0$, $0 \leqslant i < m-1$, from (24) we have

$$\Big(\beta(0,t) \otimes I_{n^{m-1}}\Big) \frac{\partial^{m-1}}{\partial x^{m-1}} x^{[m-1]} = 0. \tag{25}$$

Note that $0 \neq \frac{\partial^{m-1}}{\partial x^{m-1}} x^{[m-1]} \in \mathbb{R}^{n^{2(m-1)}}$. From (25) and $\beta(x,t) = \gamma(x,t)\, E_{n^{m-1}}$, we obtain $\beta(0,t) = 0$.

Consider each component of $\beta(x,t) := [b_1(x,t), b_2(x,t), \ \cdots, \ b_{n^{m-1}}(x,t)]$. Since the lemma holds when $m = 1$, there exist row vectors $\alpha_i(x,t) \in \mathbb{R}^n$ such that

$$b_i(x,t) = \alpha_i(x,t)x, \ \ i = 1, 2, ..., n^{m-1}.$$

Let

$$\alpha(x,t) = \Big[\alpha_1(x,t), \ \alpha_2(x,t), \cdots, \alpha_{n^{m-1}}(x,t)\Big]^T \in \mathbb{R}^{n^m},$$

then, (23) becomes

$$f(x,t) = \alpha^T(x,t)\Big(x \otimes x^{[m-1]}\Big) = \alpha^T(x,t)\, x^{[m]},$$

which implies that the lemma holds for case m.

According to the mathematical induction, the lemma holds for arbitrary m.

Corollary 1. Assume that $X(x,t) \in \mathbb{R}^p$ ($x \in \mathbb{R}^n, \ t \in \mathbb{R}^+$) is a vector field. If $X(x,t)$ has continuous mth-order partial derivatives with respect to x and satisfies $\frac{\partial^{s-1}}{\partial x^{s-1}} X(0,t) = 0$, $s = 1, 2, ..., m$, then there exists a matrix $A(x,t) \in \mathbb{R}^{p \times n^m}$ such that

$$X(x,t) = A(x,t)\, x^{[m]}. \tag{26}$$

Proof. It follows immediately from Lemma 2. QED

In the following, we apply the results above to study the generalized Hamiltonian realization for system (1).

Theorem 1. Assume that system (1) is smooth and $z = \Phi(x,t)$ with $\Phi(0,t) = 0$ is a diffeomorphism. Then, system (1) has a generalized Hamiltonian realization with Hamiltonian function $H(x,t) = \frac{1}{2}\sum_{i=1}^{n} \Phi_i^2(x,t)$ if and only if (iff) $f(0,t) = 0$, where $\Phi_i(x,t)$ is the ith component of $\Phi(x,t)$, $i = 1, 2, ..., n$.

Proof. First, we show that $f(0,t)=0$ implies that system (1) has a GHR with Hamiltonian function $H(x,t)=\frac{1}{2}\sum_{i=1}^{n}\Phi_i^2(x,t)$.

Assume that $f(0,t)=0$, then it follows from Corollary 1 that

$$\begin{aligned} f(x,t) &= f(\Phi^{-1}(z,t),t) = f_z(z,t) \\ &= A_z(z,t)z = A_z(\Phi(x,t),t)\Phi(x,t) := A(x,t)\Phi(x,t). \end{aligned} \tag{27}$$

Let $H(x,t)=\frac{1}{2}\sum_{i=1}^{n}\Phi_i^2(x,t)$, then we can obtain

$$\frac{\partial H(x,t)}{\partial x} = \Big[\frac{\partial \Phi(x,t)}{\partial x}\Big]_{n\times n}^{T}\Phi(x,t). \tag{28}$$

Since $z=\Phi(x,t)$ is a diffeomorphism, $\frac{\partial \Phi(x,t)}{\partial x}$ is nonsingular. Thus, from (27) and (28), system (1) has a generalized Hamiltonian realization as follows:

$$\dot{x} = T(x,t)\frac{\partial H(x,t)}{\partial x},\ \ T(x,t) := A(x,t)\Big[\frac{\partial \Phi(x,t)}{\partial x}\Big]^{-T}.$$

Next, we show that system (1) has a GHR with Hamiltonian function $H(x,t)=\frac{1}{2}\sum_{i=1}^{n}\Phi_i^2(x,t)$ implies that $f(0,t)=0$.

Assume that system (1) is realized as

$$\dot{x} = T(x,t)\frac{\partial H(x,t)}{\partial x},\ \ H(x,t) = \frac{1}{2}\sum_{i=1}^{n}\Phi_i^2(x,t).$$

Then, eq. (28) still holds. From (28) and $\Phi(0,t)=0$, we know that $\frac{\partial H(0,t)}{\partial x}=0$, from which it follows that $f(0,t)=0$.

Remark 3. Since system (1) can always be made to satisfy $f(0,t)=0$ under a suitable coordinate transformation, we can conclude, from Theorem 1, that every smooth system can be expressed as a Hamiltonian system under, if it is needed, certain coordinate transformation.

Let $J_f(x,t)$ denote the Jacobian matrix of $f(x,t)$, i.e., $J_f(x,t)=\left[\frac{\partial f_i}{\partial x_j}\right]\in\mathbb{R}^{n\times n}$. When $J_f(x,t)$ is nonsingular, $z:=f(x,t)$ can be taken as a diffeomorphism. Motivated by Theorem 1, system (1) should have a GHR with Hamiltonian function $H(x,t)=\frac{1}{2}\sum_{i=1}^{n}f_i^2(x,t)$, which leads to the following result.

Corollary 2. If the Jacobian matrix $J_f(x,t)$ is nonsingular, then system (1) has a generalized Hamiltonian realization as follows:

$$\dot{x} = J_f^{-T}(x,t)\frac{\partial H(x,t)}{\partial x}, \tag{29}$$

where

$$H(x,t) = \frac{1}{2}\sum_{i=1}^{n} f_i^2(x,t), \tag{30}$$

and $f_i(x,t)$ is the ith component of $f(x,t)$, $i=1,2,...,n$.

Proof. A straightforward computation shows that

$$J_f^T(x,t)f(x,t) = \begin{bmatrix} \displaystyle\sum_{i=1}^{n} f_i(x,t)\frac{\partial f_i(x,t)}{\partial x_1} \\ \vdots \\ \displaystyle\sum_{i=1}^{n} f_i(x,t)\frac{\partial f_i(x,t)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^n.$$

From (30), we obtain

$$\frac{\partial H(x,t)}{\partial x}=\begin{bmatrix}\sum_{i=1}^{n} f_i(x,t)\frac{\partial f_i(x,t)}{\partial x_1}\\ \vdots\\ \sum_{i=1}^{n} f_i(x,t)\frac{\partial f_i(x,t)}{\partial x_n}\end{bmatrix}.$$

Accordingly, we have

$$J_f^T(x,t)f(x,t)=\frac{\partial H(x,t)}{\partial x},\tag{31}$$

from which and the invertibility of $J_f(x,t)$, Corollary 2 follows immediately.

Remark 4. From (30), it can be seen that Hamiltonian function $H(x,t)$ in (30) is positive except for $H(x,t)=0$ at the equilibrium of system (1). Therefore, $H(x,t)$ is positive definite when the system's equilibrium is isolated.

Remark 5. $J_f^T(x,t)$ can be taken as a structural matrix. In fact, if we assume that $z=\psi(x)$ is an arbitrary coordinate transformation, then $\dot{z}=\frac{\partial\psi}{\partial x}\dot{x}$, $\frac{\partial H(x,t)}{\partial x}=(\frac{\partial\psi}{\partial x})^T\frac{\partial H(\psi^{-1}(z),t)}{\partial z}$, from which and (31), we obtain

$$\Big(\frac{\partial\psi}{\partial x}\Big)^{-T}J_f^T(x,t)\Big(\frac{\partial\psi}{\partial x}\Big)^{-1}\dot{z}=\frac{\partial H(\psi^{-1}(z),\ t)}{\partial z}.\tag{32}$$

Eq. (32) means that $J_f^T(x,t)$ is consistent with the changing law of structure matrices under coordinate transformations. Thus, $J_f^T(x,t)$ (or $J_f^{-T}(x,t)$, if $J_f^T(x,t)$ is nonsingular) can be chosen as a structural matrix.

In the following, we study the GHR of system (1) with $J_f(x,t)$ singular, and propose a method to handle this case.

Lemma 3. Assume that $X(x,t)$ with $X(0,t)=0$ is an n-dimensional vector field. If the Jacobian matrix $J_X(x,t)$ is singular and has a nonsingular main diagonal block, then there exists a matrix $T_X(x,t)\in\mathbb{R}^{n\times n}$ and a vector field $\bar{X}(x,t)$ with $J_{\bar{X}}(x,t)$ nonsingular such that

$$X(x,t)=T_X(x,t)\bar{X}(x,t).\tag{33}$$

Proof. Without loss of generality, we assume that Rank $\{J_X(x,t)\}=k<n$ and $\frac{\partial(X_1,\ldots,X_k)}{\partial(x_1,\ldots,x_k)}$ is a nonsingular main diagonal block, where X_i denotes the ith component of $X(x,t)$.

Denote

$$X_I(x,t)=\begin{bmatrix}X_1(x,t)\\ \vdots\\ X_k(x,t)\end{bmatrix},\ X_{II}(x,t)=\begin{bmatrix}X_{k+1}(x,t)\\ \vdots\\ X_n(x,t)\end{bmatrix},\ x_I=\begin{bmatrix}x_1\\ \vdots\\ x_k\end{bmatrix},\ x_{II}=\begin{bmatrix}x_{k+1}\\ \vdots\\ x_n\end{bmatrix}.\tag{34}$$

Since $X_{II}(0,t)=0$, it can be seen from Corollary 1 that there is a matrix $A(x,t)\in\mathbb{R}^{(n-k)\times n}$ such that

$$X_{II}(x,t)=A(x,t)x=A_1(x,t)x_I+A_2(x,t)x_{II},\tag{35}$$

where $A(x,t)=[A_1(x,t),A_2(x,t)]$, $A_1(x,t)\in\mathbb{R}^{(n-k)\times k}$, and $A_2(x,t)\in\mathbb{R}^{(n-k)\times(n-k)}$. On the other hand, $X_I=X_I(x_I,x_{II};t)$, $X_I(0,0;t)=0$ and $\frac{\partial X_I}{\partial x_I}$ is nonsingular. It follows from the implicit function theorem that there exists a function ψ such that

$$x_I=\psi(X_I,x_{II};t),\ \ \psi(0,0;t)=0.\tag{36}$$

From Corollary 1 again, there is a matrix $B(x,t) = [B_1(x,t), B_2(x,t)] \in \mathbb{R}^{k\times n}$ such that (36) can be expressed as

$$x_{_I} = B(x,t)\begin{bmatrix} X_{_I} \\ x_{_{II}} \end{bmatrix} = B_1(x,t)X_{_I} + B_2(x,t)x_{_{II}}. \tag{37}$$

Substituting (37) into (35) gives

$$X_{_{II}} = A_1(x,t)B_1(x,t)X_{_I} + \Big[A_1(x,t)B_2(x,t) + A_2(x,t)\Big]x_{_{II}}.$$

Therefore, we have

$$\begin{bmatrix} X_{_I} \\ X_{_{II}} \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ A_1(x,t)B_1(x,t) & A_1(x,t)B_2(x,t) + A_2(x,t) \end{bmatrix}\begin{bmatrix} X_{_I} \\ x_{_{II}} \end{bmatrix}.$$

Let

$$T_{_X}(x,t) = \begin{bmatrix} I_k & 0 \\ A_1(x,t)B_1(x,t) & A_1(x,t)B_2(x,t) + A_2(x,t) \end{bmatrix}, \quad \bar{X}(x,t) = \begin{bmatrix} X_{_I} \\ x_{_{II}} \end{bmatrix}. \tag{38}$$

Then $X(x,t) = T_{_X}(x,t)\bar{X}(x,t)$, and

$$J_{\bar{X}}(x,t) = \begin{bmatrix} \frac{\partial X_{_I}}{\partial x_{_I}} & \frac{\partial X_{_I}}{\partial x_{_{II}}} \\ 0 & I_{n-k} \end{bmatrix}$$

is nonsingular. Thus, Lemma 3 holds. QED

Now, consider system (1) with $f(0,t) = 0$. Assume that the Jacobian matrix $J_f(x,t)$ is singular and has a nonsingular main diagonal block. From Lemma 3, there exists a matrix $T_f(x,t) \in \mathbb{R}^{n\times n}$ and a vector field $\bar{f}(x,t) \in \mathbb{R}^n$, with $J_{\bar{f}}(x,t)$ nonsingular, such that

$$f(x,t) = T_f(x,t)\bar{f}(x,t). \tag{39}$$

From Corollary 2, system (1) has a GHR as follows:

$$\dot{x} = T_f(x,t)J_{\bar{f}}^{-T}(x,t)\frac{\partial H(x,t)}{\partial x}, \tag{40}$$

where $H(x,t) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\bar{f}_i^2(x,t)$, $\bar{f}(x,t) = [\bar{f}_1(x,t), ..., \bar{f}_n(x,t)]^T$.

Theorem 2. Assume that the Jacobian matrix $J_f(x,t)$ is singular and has a nonsingular main diagonal block. If $f(0,t) = 0$, then system (1) has a GHR given by (40).

Remark 6. From the proof of Lemma 3, it can be seen that the proof itself gives a practical algorithm to find $T_f(x,t)$ and $\bar{f}(x,t)$ already. Using the algorithm, we can easily obtain the GHR (40).

Example 1. Consider the following system

$$\dot{x} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -x_1 - x_2\sin t + x_3 \\ -x_2 + x_1\sin t - x_3 \\ -x_1 - x_2 + (x_1 - x_2)\sin t \end{bmatrix}, \quad x \in \mathbb{R}^3, \ \ t \in \mathbb{R}^+. \tag{41}$$

A straightforward computation shows that Rank $[J_f(x,t)] = 2 < 3$ and $\frac{\partial(f_1,f_2)}{\partial(x_1,x_2)}$ is nonsingular. In this example, $X_{I}=[f_1,\, f_2]^T$, $X_{II}=f_3$, $x_{I}=[x_1,\, x_2]^T$, and $x_{II}=x_3$. From (41), we have

$$f_3=(-1+\sin t)x_1+(-1-\sin t)x_2=A_1(x,t)\begin{bmatrix}x_1\\x_2\end{bmatrix}+A_2(x,t)x_3, \tag{42}$$

where $A_1(x,t)=\Big[-1+\sin t,\ -1-\sin t\Big]$ and $A_2(x,t)=0$. From

$$\begin{cases} f_1=-x_1-x_2\sin t+x_3,\\ f_2=-x_2+x_1\sin t-x_3,\end{cases}$$

we obtain

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=B_1(x,t)\begin{bmatrix}f_1\\f_2\end{bmatrix}+B_2(x,t)x_3, \tag{43}$$

where

$$B_1(x,t)=\frac{1}{1+\sin^2 t}\begin{bmatrix}-1 & \sin t\\ -\sin t & -1\end{bmatrix},\quad B_2(x,t)=\frac{1}{1+\sin^2 t}\begin{bmatrix}1+\sin t\\ -1+\sin t\end{bmatrix}.$$

It follows from (42) and (43) that

$$f_3=A_1B_1\begin{bmatrix}f_1\\f_2\end{bmatrix}+(A_1B_2+A_2)x_3=(1,1)\begin{bmatrix}f_1\\f_2\end{bmatrix}+0\cdot x_3.$$

Thus,

$$\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\\ \begin{bmatrix}1&1\end{bmatrix} & 0\end{bmatrix}\begin{bmatrix}f_1\\f_2\\x_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\1&1&0\end{bmatrix}\begin{bmatrix}f_1\\f_2\\x_3\end{bmatrix}:=T_f(x,t)\bar f(x,t). \tag{44}$$

From (40), system (41) has a GHR as follows:

$$\begin{aligned}\dot x &= T_f(x,t)J_{\bar f}^{-T}(x,t)\frac{\partial H(x,t)}{\partial x}\\
&=\begin{bmatrix}1&0&0\\0&1&0\\1&1&0\end{bmatrix}\begin{bmatrix}-1 & -\sin t & 1\\ \sin t & -1 & -1\\ 0 & 0 & 1\end{bmatrix}^{-T}\frac{\partial H(x,t)}{\partial x}\\
&=\begin{bmatrix}1&0&0\\0&1&0\\1&1&0\end{bmatrix}\begin{bmatrix}\frac{-1}{1+\sin^2 t} & \frac{-\sin t}{1+\sin^2 t} & 0\\ \frac{\sin t}{1+\sin^2 t} & \frac{-1}{1+\sin^2 t} & 0\\ \frac{1+\sin t}{1+\sin^2 t} & \frac{-1+\sin t}{1+\sin^2 t} & 1\end{bmatrix}\frac{\partial H(x,t)}{\partial x},\end{aligned}$$

that is,

$$\dot x=\frac{1}{1+\sin^2 t}\begin{bmatrix}-1 & -\sin t & 0\\ \sin t & -1 & 0\\ -1+\sin t & -1-\sin t & 0\end{bmatrix}\frac{\partial H(x,t)}{\partial x}, \tag{45}$$

where

$$H(x,t) = \frac{1}{2}(f_1^2 + f_2^2) + \frac{1}{2}x_3^2. \tag{46}$$

4 Dissipative Hamiltonian realization

Dissipative Hamiltonian realization is a very important case of GHR, and has many applications to various practical control problems. This section investigates the dissipative Hamiltonian realization of system (1). First, we present a sufficient condition, called a Krasovskii-like condition, and then provide a sufficient and necessary condition for the realization.

Theorem 3. If there exists a constant positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that

$$Q(x,t) := PJ_f(x,t) + J_f^T(x,t)P < 0, \tag{47}$$

then system (1) has a dissipative Hamiltonian realization as follows:

$$\dot{x} = \Big[J(x,t) - R(x,t)\Big]\frac{\partial H(x,t)}{\partial x}, \tag{48}$$

where $J(x,t) \in \mathbb{R}^{n\times n}$ is skew-symmetric, $R(x,t) \in \mathbb{R}^{n\times n}$ is positive semi-definite and $H(x,t)$ is positive definite.

Proof. We first show that (47) implies that $PJ_f(x,t)$ is nonsingular. In fact, if $PJ_f(x,t)$ is singular, there exists a vector $0 \neq \alpha \in \mathbb{R}^n$ such that $[PJ_f(x,t)]\alpha = 0$, from which we obtain $\alpha^T[PJ_f(x,t)]\alpha = 0$. Thus,

$$\alpha^T[PJ_f(x,t)]\alpha + \alpha^T[J_f^T(x,t)P]\alpha = \alpha^T[PJ_f(x,t) + J_f^T(x,t)P]\alpha = 0, \tag{49}$$

which contradicts the fact that $Q(x,t)$ is negative definite. Therefore, $PJ_f(x,t)$ is nonsingular, which implies that $J_f(x,t)$ is also nonsingular.

Choosing $H(x,t) = \frac{1}{2}f^T(x,t)Pf(x,t)$, then, similar to Corollary 2, we can show that system (1) has a GHR as follows:

$$\dot{x} = P^{-1}J_f^{-T}(x,t)\frac{\partial H(x,t)}{\partial x}. \tag{50}$$

On the other hand, we have the following equivalent relation

$$\begin{aligned}
Q(x,t) < 0 &\Longleftrightarrow P^{-1}Q(x,t)P^{-T} < 0 \\
&\Longleftrightarrow P^{-1}\Big[PJ_f(x,t) + J_f^T(x,t)P\Big]P^{-T} < 0 \\
&\Longleftrightarrow J_f(x,t)P^{-T} + P^{-1}J_f^T(x,t) < 0 \\
&\Longleftrightarrow J_f^{-1}(x,t)\Big[J_f(x,t)P^{-T} + P^{-1}J_f^T(x,t)\Big]J_f^{-T}(x,t) < 0 \\
&\Longleftrightarrow P^{-T}J_f^{-T}(x,t) + J_f^{-1}(x,t)P^{-1} < 0 \\
&\Longleftrightarrow P^{-1}J_f^{-T}(x,t) + J_f^{-1}(x,t)P^{-T} < 0.
\end{aligned}$$

Thus, $P^{-1}J_f^{-T}(x,t)$ can be expressed as

$$P^{-1}J_f^{-T}(x,t) = J(x,t) - R(x,t), \tag{51}$$

where

$$J(x,t) = \frac{1}{2}\Big[P^{-1}J_f^{-T}(x,t) - J_f^{-1}(x,t)P^{-T}\Big] = -J^T(x,t),$$

$$R(x,t)=\frac{1}{2}\Big\{-\Big[P^{-1}J_f^{-T}(x,t)+J_f^{-1}(x,t)P^{-T}\Big]\Big\}>0.$$

Therefore, (50) is a dissipative Hamiltonian realization, which means that Theorem 3 holds.

Theorem 4. Assume that $V(x,t)$ is a positive definite function satisfying $\nabla V(x,t)|_{x\neq 0}\neq 0$. Then, system (1) with $f(0,t)=0$ has a dissipative Hamiltonian realization with $V(x,t)$ as its Hamiltonian function iff $L_{f(x,t)}V(x,t)\leqslant 0$.

Proof. First, we show that the system has a dissipative Hamiltonian realization with $V(x,t)$ as its Hamiltonian function implies that $L_{f(x,t)}V(x,t)\leqslant 0$.

Assume that system (1) has a dissipative Hamiltonian realization as follows:

$$\dot{x}=\Big[J(x,t)-R(x,t)\Big]\frac{\partial V(x,t)}{\partial x}, \tag{52}$$

where $J(x,t)\in\mathbb{R}^{n\times n}$ is skew-symmetric and $0\leqslant R(x,t)\in\mathbb{R}^{n\times n}$. Then,

$$\begin{aligned}L_{f(x,t)}V(x,t)&=\frac{\partial V^T(x,t)}{\partial x}f(x,t)=\frac{\partial V^T(x,t)}{\partial x}\Big[J(x,t)-R(x,t)\Big]\frac{\partial V(x,t)}{\partial x}\\&=-\frac{\partial V^T(x,t)}{\partial x}R(x,t)\frac{\partial V(x,t)}{\partial x}\leqslant 0.\end{aligned}$$

Next, we show that $L_{f(x,t)}V(x,t)\leqslant 0$ implies that the system has a dissipative Hamiltonian realization with $V(x,t)$ as its Hamiltonian function.

Assume that $L_{f(x,t)}V(x,t)\leqslant 0$. Then, since $\nabla V(x,t)|_{x\neq 0}\neq 0$, we can construct the following matrices

$$R(x,t)=\begin{cases}-\dfrac{L_{f(x,t)}V(x,t)}{||\nabla V(x,t)||^2}I_n & x\neq 0,\\ 0 & x=0,\end{cases} \tag{53}$$

$$J(x,t)=\begin{cases}\dfrac{1}{||\nabla V(x,t)||^2}\Big[\tilde{f}(x,t)\dfrac{\partial V^T(x,t)}{\partial x}-\dfrac{\partial V(x,t)}{\partial x}\tilde{f}^T(x,t)\Big] & x\neq 0,\\ 0 & x=0,\end{cases} \tag{54}$$

where

$$\tilde{f}(x,t)=f(x,t)-\hat{f}(x,t),\quad \hat{f}(x,t)=\frac{L_{f(x,t)}V(x,t)}{||\nabla V(x,t)||^2}\nabla V(x,t)\ (x\neq 0). \tag{55}$$

As $L_{f(x,t)}V(x,t)\leqslant 0$, it is easy to see that $R(x,t)\geqslant 0$ and $J(x,t)$ is skew-symmetric.

From (55), we obtain

$$\begin{aligned}L_{\tilde{f}(x,t)}V(x,t)&=L_{(f(x,t)-\hat{f}(x,t))}V(x,t)=L_{f(x,t)}V(x,t)-L_{\hat{f}(x,t)}V(x,t)\\&=L_{f(x,t)}V(x,t)-\nabla^T V(x,t)\nabla V(x,t)\frac{L_{f(x,t)}V(x,t)}{||\nabla V(x,t)||^2}=0,\end{aligned}$$

from which, we know that when $x\neq 0$,

$$\begin{aligned}J(x,t)\frac{\partial V(x,t)}{\partial x}&=\frac{1}{||\nabla V(x,t)||^2}[\tilde{f}(x,t)\frac{\partial V^T(x,t)}{\partial x}-\frac{\partial V(x)}{\partial x}\tilde{f}^T(x,t)]\frac{\partial V(x,t)}{\partial x}\\&=\frac{1}{||\nabla V(x,t)||^2}\tilde{f}(x,t)\frac{\partial V^T(x,t)}{\partial x}\frac{\partial V(x,t)}{\partial x}-\frac{1}{||\nabla V(x,t)||^2}\frac{\partial V(x,t)}{\partial x}\tilde{f}^T(x,t)\frac{\partial V(x,t)}{\partial x}\\&=\frac{1}{||\nabla V(x,t)||^2}\tilde{f}(x,t)||\nabla V(x,t)||^2-\frac{1}{||\nabla V(x,t)||^2}\frac{\partial V(x,t)}{\partial x}L_{\tilde{f}}V(x,t)=\tilde{f}(x,t).\end{aligned}$$

Thus, when $x \neq 0$,

$$
\begin{aligned}
f(x,t) = \tilde{f}(x,t) + \hat{f}(x,t) &= J(x,t)\frac{\partial V(x,t)}{\partial x} - R(x,t)\frac{\partial V(x,t)}{\partial x} \\
&= \Big[J(x,t) - R(x,t)\Big]\frac{\partial V(x,t)}{\partial x}.
\end{aligned} \tag{56}
$$

Note that (56) still holds when $x = 0$. The proof of the sufficiency is completed.

Remark 7. Note that $\dot{V} = L_{f(x,t)}V(x,t) + \frac{\partial V(x,t)}{\partial t}$. Thus, if we add a condition $\frac{\partial V(x,t)}{\partial t} \leqslant 0$ in Theorem 4, the theorem implies that a system is stable is equivalent to that the system has a dissipative Hamiltonian realization, which is very useful in the system control designs.

5 Conclusion

We have investigated the Hamiltonian realization for TVN systems, and developed several new methods to deal with the problem. It has been shown that every smooth TVN system can be expressed as a generalized Hamiltonian one if the origin is the system's equilibrium. If the Jacobian matrix of a TVN is nonsingular, the system has a GHR whose structural matrix and Hamiltonian function are given explicitly. For the case that the Jacobian matrix is singular, we provided a constructive decomposition method for the GHR. Furthermore, we obtained some sufficient (necessary and sufficient) conditions for dissipative Hamiltonian realization. The results presented in the paper not only provide solid theoretical frameworks, but also give several feasible algorithms for practical Hamiltonian realization.

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