# Stability and Stabilization of Block-cascading Switched Linear Systems

Ya-Hong Zhu*, Dai-Zhan Cheng
Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, PRC

**Abstract:** The main purpose of this paper is to investigate the problem of quadratic stability and stabilization in switched linear systems using reducible Lie algebra. First, we investigate the structure of all real invariant subspaces for a given linear system. The result is then used to provide a comparable cascading form for switching models. Using the common cascading form, a common quadratic Lyapunov function is (QLFs) is explored by finding common QLFs of diagonal blocks. In addition, a cascading Quaker Lemma is proved. Combining it with stability results, the problem of feedback stabilization for a class of switched linear systems is solved.

**Keywords:** Switching linear system, quadratic stability, common quadratic Lyapunov function, reducible Lie algebra, cascading Quaker Lemma.

## 1 Introduction

In recent years the investigation of switched systems has attracted more and more attention, we refer to [1] and the references therein. This is due to the fact that a wide variety of natural and engineering systems are inherently multi-model[2~4]. One important topic in investigating switched systems is stability. To solve this problem, an unnecessary but natural way is to find a common Lyapunov function for all switching models. Such a Lyapunov function suffices for the stability of systems under arbitrary switchings. For linear switching models, a quadratic Lyapunov function (QLF) plays an important role.

**Definition 1.** Consider a switched linear system

$$\dot{x} = A_{\sigma(t)}x, \quad x \in \mathbb{R}^n \tag{1}$$

where $\sigma(t) : [0 \quad \infty) \to \Lambda$ is a right continuous measurable mapping, and $\Lambda = \{1, 2, \cdots, N\}$. (1) is said to be quadratically stable if there is a positive definite matrix $P > 0$ such that

$$PA_i + A_i^{\mathrm{T}}P < 0, \quad i = 1, \cdots, N. \tag{2}$$

If (2) holds, we say that $A_i$, $i = 1, \cdots, N$ share a common QLF. The problem of finding a common QLF has been studied for a long time. Many different methods have been used to solve it[5~7]. Recently, a necessary and sufficient condition for the existence of a common QLF for a set of stable matrices was presented in [8]. In fact, [8] provides a numerical method for verifying such an existence, and when $n = 2$ this becomes an easily verifiable necessary and sufficient condition.

In general, stabilization in switched systems is an even harder problem. We state it formally as follows:

**Definition 2.** Consider a switched linear system

$$\dot{x} = A_{\sigma(t)}x + B_{\sigma(t)}u_{\sigma(t)}, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{3}$$

where $\sigma(t) : [0 \quad \infty) \to \Lambda$ is a right continuous measurable mapping, and $\Lambda = \{1, 2, \cdots, N\}$. The quadratic stabilization problem is: find the feedback controls $u_i = K_i x$ and $i = 1, \cdots, N$, and a positive definite matrix $P > 0$, such that $\tilde{A}_i = A_i + B_i K_i$, $i = 1, \cdots, N$ share a common QLF $x^{\mathrm{T}}Px$.

For planar switched linear systems, a necessary and sufficient condition was given in [9]. For $n > 2$, the problem remains open.

A useful result, which may simplify the quadratic stability and stabilization problem significantly, is the following:

**Theorem 1**[8]**.** Let $A_i$, $i = 1, \cdots, N$ be a set of Hurwitz matrices with the same block upper triangular structure, i.e.,

$$A_i = \begin{pmatrix} A_i^{11} & A_i^{12} & \cdots & A_i^{1s} \\ 0 & A_i^{22} & \cdots & A_i^{2s} \\ \vdots & & & \\ 0 & 0 & \cdots & A_i^{ss} \end{pmatrix}, \quad i = 1, \cdots, N \tag{4}$$

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*Corresponding author. *E-mail address*: yhzhu@amss.ac.cn

where $A_i^{kk}$, $k = 1, \cdots, s$ are $n_k \times n_k$ matrices. Then $A_i$ share a common QLF, iff $A_i^{kk}$ share a common QLF for all $k = 1, \cdots, s$.

The purpose of this paper is to investigate when the switched linear system (1) or closed-loop of (3) can have the cascading form (4). Then Theorem 1 can be used to test the quadratic stability of system (1), or the stabilization of system (3).

This paper is organized as follows. Section 2 considers the relationship between cascading form and invariant subspaces. Section 3 provides a complete description of the invariant subspaces of a given linear system. Section 4 investigates quadratic stability by using cascading realization. Section 5 considers the quadratic stabilization problem by using a cascading form. Section 6 provides a conclusion.

## 2 Cascading form vs. invariant subspace

In this section, we show that for a given matrix its cascading form is closely related to its invariant subspaces, which are defined as follows.

**Definition 3**[10] **.** 1. For a linear system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n \tag{5}$$

a subspace $V \subset \mathbb{R}^n$ is called $A$-invariant if

$$AV \subset V.$$

2. For a linear control system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{6}$$

a subspace $V \subset \mathbb{R}^n$ is called $(A, B)$-invariant, if

$$AV \subset V + \mathcal{B} \tag{7}$$

where $B = (b_1, \cdots, b_m)$ and $\mathcal{B}$ is the subspace spanned by $\{b_1, \cdots, b_m\}$.

When we consider the feedback case, the following fact is important:

**Lemma 1.** (Quaker Lemma)[10] A subspace $V$ is $(A, B)$-invariant, iff there exists a feedback control $u = Kx$, such that

$$(A + BK)V \subset V. \tag{8}$$

Next, we consider when $A_i$ in system (1) can be converted into cascading form (4) simultaneously.

**Proposition 1.** $A_i$, $i \in \Lambda$ can be converted into cascading form (4), iff there exists

$$0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_{s-1} \subsetneqq V_s = R^n \tag{9}$$

to which all $A_i$ are invariant.

All proof in this paper is in the Appendix.

## 3 Structure of invariant subspaces

In this section, we investigate the structure of all real invariant subspaces of given matrix $A$.

Suppose the minimum polynomial of a matrix $A$ is

$$\begin{aligned} m(\lambda) = {} & (\lambda - \lambda_1)^{l_1}(\lambda - \lambda_2)^{l_2} \cdots (\lambda - \lambda_k)^{l_k} \times \\ & (\lambda - \alpha_1 + \beta_1 i)^{c_1}(\lambda - \alpha_1 - \beta_1 i)^{c_1} \cdots \\ & (\lambda - \alpha_s + \beta_s i)^{c_s}(\lambda - \alpha_s - \beta_s i)^{c_s} \end{aligned} \tag{10}$$

where $\lambda_t$, $t = 1 \cdots k$, $\alpha_j \pm \beta_j i$, $j = 1, \cdots, s$ are two groups of distinct real and complex eigenvalues of $A$ respectively.

Throughout this paper, we denote $\gamma_j = \alpha_j + \beta_j i$, $\bar{\gamma}_j = \alpha_j - \beta_j i$, and $B_j = A^2 - (\gamma_j + \bar{\gamma}_j)A + |\gamma_j|^2 I = A^2 - 2\alpha_j A + (\alpha_j^2 + \beta_j^2)I$, $j = 1, \cdots, s$.

First, we need the following two decomposition results; they are well known facts in Linear Algebra.

**Lemma 2**[11]**.** The whole space $\mathbb{R}^n$ can be decomposed into several kernel subspaces as

$$\begin{aligned} R^n = {} & ker(A - \lambda_1 I)^{l_1} \oplus \cdots \oplus ker(A - \lambda_k I)^{l_k} \oplus \\ & ker(B_1)^{c_1} \oplus \cdots \oplus ker(B_s)^{c_s} := \\ & V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_s \end{aligned} \tag{11}$$

where $V_t = ker(A - \lambda_t I)^{l_t}$, $t = 1, \cdots, k$, and $W_j = ker(B_j)^{c_j}$, $j = 1, \cdots, s$.

**Lemma 3**[11]**.** If a subspace $H \subset \mathbb{R}^n$ is A-invariant, then

$$H = (H \cap V_1) \oplus \cdots \oplus (H \cap V_k) \oplus (H \cap W_1) \oplus \cdots \oplus (H \cap W_s)$$

where $V_t$ and $W_j$ are as in (11).

Now we are ready to consider the set of all real invariant subspaces of $A$.

First, we consider the case in which matrix $A$ has a unique real eigenvalue.

**Assumption 1.** Matrix $A$ has a unique real eigenvalue, and under basis $e_1^1, \cdots, e_1^{s_1}, \cdots, e_m^1, \cdots, e_m^{s_m}$ $A$ can be expressed in Jordan canonical form, $T^{-1}AT = diag\{J_1, \cdots, J_m\}$, with

$$J_i(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}_{s_i \times s_i}$$

$$i = 1, \cdots, m, \ \sum_{i=1}^{m} s_i = n \tag{12}$$

where $T = [e_1^1, \cdots, e_1^{s_1}, \cdots, e_m^1, \cdots, e_m^{s_m}]$.

**Definition 4.** For a vector $\xi \in \mathbb{R}^n$, the smallest subspace $Z(\xi, A)$ containing $\xi$ and $A$ invariant is called the $A$-cyclic subspace of $\xi$.

**Remark 1.** Let $t > 0$ be the smallest integer such that

$$Span\{\xi, A\xi, \cdots, A^{t-1}\xi\} = Span\{\xi, A\xi, \cdots, A^t\xi\}$$

Then subspace $Z(\xi, A) = Span\{\xi, A\xi, \cdots, A^{t-1}\xi\}$ and $\dim(Z(\xi, A)) = t$. The following lemma is obvious (or a well known fact).

**Lemma 4.** If subspace $H \subset \mathbb{R}^n$ is $A$-invariant, then H can be decomposed into the direct sum of several $A$-cyclic subspaces.

Now we are ready to find any $t$-dimensional $A$-invariant subspace, say $H$. Recalling matrix (12), we denote: $s = \max\{s_i | i = 1, \cdots, m\}$, $C = A - \lambda I$. Using $Z_+$ for the set of positive integers, we define

$$K_t = \Big\{k = (k_1, k_2, \cdots, k_\tau) \in Z_+^\tau \Big| \tau \leqslant m, \sum_{j=1}^{\tau} k_j = t, 0 < k_j \leqslant k_{j+1} \leqslant s\Big\}.$$

For each $k \in K_t$, our purpose is to find $k_j$, $j = 1, \cdots, \tau$, dimensional linearly independent $A$-invariant subspaces such that $H$ is a direct sum of these linearly independent subspaces.

Construct a set of vectors $\xi_j \in \mathbb{R}^n$

$$\xi_j = \sum_{i=1}^{m} \sum_{l=1}^{k_j} c_{i,l}^j e_i^l, \quad j = 1, \cdots, \tau \tag{13}$$

$c_{i,l}^j$, $i = 1, \cdots, m$, $l = 1, \cdots, k_j$, are parameters with at least one $i$ such that $c_{i,k_j}^j \neq 0$. Then, by straightforward computing we know that

$$\xi_j \in ker(C^{k_j}) \backslash ker(C^{k_j - 1}).$$

In addition, one sees easily that

$$Z(\xi_j, C) := Span\{\xi_j, C\xi_j, \cdots, C^{k_j - 1}\xi_j\} = Z(\xi_j, A). \tag{14}$$

Then we have the following result, which shows $H$ can be constructed by using such blocks.

**Proposition 2.** Assume Assumption 1 holds for $A$. Then for each $k \in K_t$, we can construct a t-dimensional $A$-invariant subspace $H \subset \mathbb{R}^n$ as a direct sum of $Z(\xi_j, C)$, that is, $H = \oplus_{j=1}^{\tau} Z(\xi_j, C)$, if $[c_{1,k_j}^j, c_{2,k_j}^j, \cdots, c_{m,k_j}^j]^{\mathrm{T}}$, $j = 1, \cdots, \tau$ are linearly independent, where $\xi_i \in \mathbb{R}^n$ is constructed as in (13), and $Z(\xi_j, C)$ is defined as in (14).

**Proposition 3.** Assume Assumption 1 holds for $A$, and $H \subset \mathbb{R}^n$ is a t-dimensional $A$-invariant subspace. Then there exists a $k \in K_t$ and a set of linearly independent vectors $[c_{1,k_j}^j, c_{2,k_j}^j, \cdots, c_{m,k_j}^j]^{\mathrm{T}}$, $j = 1, \cdots, \tau$, such that $H = \oplus_{j=1}^{\tau} Z(\xi_j, C)$, where $\xi_j$ is defined as in (13), and $Z(\xi_j, C)$ is defined as in (14) and $\tau$ is a positive number determined by $H$ itself.

Proposition 3 shows that the searching procedure proposed in Proposition 2 can be used to find all $A$-invariant subspaces.

Second, we consider the case in which matrix $A$ has a unique pair of complex eigenvalues.

**Assumption 2.** Matrix $A$ has a unique pair of conjugate complex eigenvalues $\alpha \pm \beta_i$ $(\gamma = \alpha + \beta_i)$, and under basis $\eta_1^1, \cdots, \eta_1^{2s_1}, \cdots, \eta_m^1, \cdots, \eta_m^{2s_m}$, $A$ can be expressed as $diag\{J_1, \cdots, J_m\}$, where

$$J_i = \begin{pmatrix} \alpha & \beta & 0 & 0 & ,\cdots, & 0 & 0 \\ -\beta & \alpha & 1 & 0 & ,\cdots, & 0 & 0 \\ 0 & 0 & \alpha & \beta & ,\cdots, & 0 & 0 \\ 0 & 0 & -\beta & \alpha & ,\cdots, & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & ,\cdots, & 0 & 0 \\ 0 & 0 & 0 & 0 & ,\cdots, & 1 & 0 \\ 0 & 0 & 0 & 0 & ,\cdots, & \alpha & \beta \\ 0 & 0 & 0 & 0 & ,\cdots, & -\beta & \alpha \end{pmatrix}_{2s_i \times 2s_i}$$

$$i = 1, \cdots, m, \quad \sum_{i=1}^{m} s_i = n. \tag{15}$$

The canonical real Jordan block as in (15) is not very convenient. We propose another canonical real Jordan block as follows, which itself is interesting:

$$T^{-1}AT = \begin{pmatrix} \alpha & \beta & 0 & 0 & \cdots & 0 & 0 \\ -\beta & \alpha & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha & \beta & \cdots & 0 & 0 \\ 0 & 0 & -\beta & \alpha & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \alpha & \beta \\ 0 & 0 & 0 & 0 & \cdots & -\beta & \alpha \end{pmatrix}_{2s_i \times 2s_i}$$

$$i = 1, \cdots, m, \quad \sum_{i=1}^{m} s_i = n. \tag{16}$$

We prove it by constructing the transfer matrix $T$. Denote

$$\gamma = \alpha + \beta i, \quad \bar{\gamma} = \alpha - \beta i, \quad B = A^2 - (\gamma + \bar{\gamma})A + |\gamma|^2 I.$$

Then we define the transformation by

$$e_i^{2s_i} = \eta_i^{2s_i}, e_i^{2s_i - 1} = \frac{(A - \alpha I)}{\beta}\eta_i 2s_i, e_i^{2s_i - 2} = \frac{B}{\beta}\eta_i^{2s_i},$$

$$e_i^{2s_i - 3} = \frac{B(A - \alpha I)}{\beta^2}\eta_i^{2s_i}, e_i 2s_i - 4 = \frac{B^2}{\beta^2}\eta_i^{2s_i}, \cdots,$$

$$e_i^2 = \frac{B^{n-1}}{\beta^{n-1}}\eta_i^{2s_i}, e_i^1 = \frac{B^{n-1}(A-\alpha I)}{\beta^n}\eta_i^{2s_i}.$$

Then it is ready to verify that using linear transformation

$$T = [e_1^1, \cdots, e_1^{2s_1}, \cdots, e_m^1, \cdots, e_m^{2s_m}]$$

we have the canonical form (16). The canonical form (16) will be used in the sequel.

Now we find any $2t$-dimensional $A$-invariant subspace, say $H$. We define $s$ and $K_t$ as before, similar to the real eigenvalue case. For each $k \in K_t$, our purpose is to find $2k_j,\ j = 1, \cdots, \tau$ dimensional linearly independent $A$-invariant subspaces, such that $H$ is a direct sum of these linearly independent subspaces.

Construct a set of vectors $\xi_j \in \mathbb{R}^{2n}$,

$$\xi_j = \sum_{i=1}^{m}\sum_{l=1}^{2k_j} c_{i,l}^j e_i^l, \quad j = 1, \cdots, \tau. \tag{17}$$

where $c_{i,l}^j$, $i = 1, \cdots, m$, $l = 1, \cdots, 2k_j$, are parameters with at least one $i$, such that $c_{i,2k_j-1}^j \neq 0$ or $c_{i,2k_j}^j \neq 0$. Then, by straightforward computing we know that

$$\xi_j \in \ker(B^{k_j})\backslash ker(B^{k_j-1}).$$

Then we have the following result, which shows that $H$ can be constructed by using this set of vectors.

$$\begin{aligned} &Z(\xi_j, A) := Span\{\xi_j, A\xi_j, \cdots, A^{2k_j-1}\xi_j\} = \\ &Span\{\xi_j, \frac{(A-\alpha I)}{\beta}\xi_j, \frac{B}{\beta}\xi_j, \frac{B(A-\alpha I)}{\beta^2}\xi_j, \frac{B^2}{\beta^2}\xi_j, \\ &\cdots, \frac{B^{k_j-1}}{\beta^{k_j-1}}\xi_j \frac{B^{k_j-1}(A-\alpha I)}{\beta^{k_j}}\xi_j\}. \end{aligned} \tag{18}$$

**Proposition 4.** Assume Assumption 2 holds for $A$. Then for each $k \in K_t$, we can construct a $2t$-dimensional real $A$-invariant subspace $H \subset R^{2n}$ as a direct sum of $Z(\xi_j, A)$, that is $H = \oplus_{j=1}^{\tau} Z(\xi_j, A)$, if $\xi_j$ is constructed as in (17), $Z(\xi_j, A)$ is defined as in (18), and the set of $2\tau$ vectors

$$\begin{aligned} &(c_{1,2k_j-1}^j, \cdots, c_{m,2k_j-1}^j, c_{1,2k_j}^j, \cdots, c_{m,2k_j}^j), \\ &(c_{1,2k_j}^j, \cdots, c_{m,2k_j}^j, -c_{1,2k_j-1}^j, \cdots, -c_{m,2k_j-1}^j), j = 1, \cdots, \tau \end{aligned}$$

are linearly independent.

**Proposition 5.** Assume Assumption 2 holds for $A$. If a $2t$-dimensional real subspace $H \subset R^{2n}$ is $A$-invariant, then there exist a $k \in K_t$ and a set of $2\tau$ linearly independent vectors

$$\begin{aligned} &(c_{1,2k_j-1}^j, \cdots, c_{m,2k_j-1}^j, c_{1,2k_j}^j, \cdots, c_{m,2k_j}^j), \\ &(c_{1,2k_j}^j, \cdots, c_{m,2k_j}^j, -c_{1,2k_j-1}^j, \cdots, -c_{m,2k_j-1}^j), j = 1, \cdots, \tau \end{aligned}$$

such that $H = \oplus_{j=1}^{\tau} Z(\xi_j, A)$, and $\xi_j$ is defined as in (17), where $\tau$ is a positive number determined by $H$ itself, and $Z(\xi_j, A)$ is defined as in (18).

The proof is similar to that in Proposition 3.

Propositions 2∼5 provide a complete description of the $t$ dimensional invariant sub-spaces of matrix $A$, whose Jordan blocks have only one real eigenvalue, or are only a pair of conjugate complex eigenvalues. Now we can put blocks of different eigenvalues together. Combining Lemma 3. and the Propositions 2∼4 yields our main result for invariant subspaces. To state it we need a new notation. Denote $Z_0$ as the set of nonnegative integers, and define an index set as

$$\begin{aligned} M_t = \{\mu = (\mu_1, \cdots, \mu_k, \mu_{k+1}, \cdots, \mu_{k+s}) \in Z_0^{k+s} | \\ \mu_1 + \cdots + \mu_k + 2(\mu_{k+1} + \cdots + \mu_{k+s}) = t\}. \end{aligned}$$

**Theorem 2.** Assume that matrix $A$ has its minimum polynomial as (10), and $\mathbb{R}^n$ is decomposed as in (11). Then a $t$-dimensional real $A$ invariant subspace $H$ is a direct sum of $H_i$, $i = 1, \cdots, k+s$. Each set of $\{H_i\}$ is determined by one $\mu \in M_t$. That is, a set of $\mu_i$-dimensional subspaces $H_i$, $i = 1, \cdots, k$ are obtained from $V_i$, and a set of $2\mu_{k+i}$-dimensional subspaces $H_{k+i}$, $i = 1, \cdots, s$ are obtained from $W_i$.

**Remark 2.** If we replace the real basis with a complex basis, using a similar approach we can find all complex invariant subspaces.

Finally, we consider the Lie algebra generated by $\{A_i\}$, and have the following.

**Proposition 6.** If switched linear system (1) has a cascading form and $\mathcal{A} = \{A_\lambda | \lambda \in \Lambda\}_{LA}$ then the new system with an enlarged set of switching models

$$\dot{z} = A_{\delta(t)} z, \quad z \in \mathbb{R}^n \tag{19}$$

also has a cascading form, where $\delta(t) : [0, \quad \infty) \longrightarrow \mathcal{A}$ is a right continuous measurable mapping.

## 4 Cascading realization vs. quadratic stability

In this section, we consider the quadratic stability of system (1) via a cascading transformation. Recall Theorem 1, and assume that $A_i^{kk}$, $i = 1, \cdots, N$, share common QLFs $P_k$, $k = 1, \cdots, s$. Then $A_i$ share a common quadratic $P$. In the following, we will first construct $P$.

Denote $V_i^k = -(P_k A_i^{kk} + (A_i^{kk})^{\mathrm{T}} P_k)$; then $V_i^k > 0$, so there exists a positive number $\varepsilon_k$ such that all the eigenvalues of $V_i^k$, $i = 1, \cdots, N$ are greater than $\varepsilon_k$. Then we denote

$$M_k = \max\left\{ \| \sum_{j=1}^{k-1} (A_i^{jk})^{\mathrm{T}} P_j^2 A_i^{jk} \| \;\middle|\; i = 1, \cdots, N \right\}$$

$$k = 2, \cdots, s \tag{20}$$

and define a set of positive numbers as:

$$0 < \delta_1 < \frac{\varepsilon_1}{s-1}, \quad 0 < \delta_k < \frac{\varepsilon_k}{2(s-k)}, \quad k = 2, \cdots, s-1. \tag{21}$$

Using (20) and (21), another sequence of positive numbers can be defined recursively as

$$\begin{cases} \mu_1 = 1 \\ \mu_k \geqslant \max\left\{\frac{2\mu_{k-1}}{\delta_{k-1}\varepsilon_k}M_k, \ \frac{\delta_k\mu_{k-1}}{\delta_{k-1}}\right\}, \ k = 2, \cdots, s-1. \\ \mu_s \geqslant \frac{\mu_{s-1}}{\delta_{s-1}\varepsilon_s}M_s \end{cases} \tag{22}$$

Using (22), we can construct a common QLF as follows.

**Lemma 5.** Assume that $A_i^{kk}$, $i = 1, \cdots, N$ share a common QLF $P_k$ for each $k = 1, \cdots, s$, and that there is a sequence $\{\mu_i \mid i = 1, \cdots, s\}$ defined through (20)$\sim$(22). Then $P = diag\{\mu_1 P_1, \cdots, \mu_s P_s\}$ is a common QLF of $A_i$, $i = 1, \cdots, N$, where $A_i^{kk}$ and $A_i$ are defined as in Theorem 1.

**Remark 3.** A simple way to choose $\{\delta_j\}$ is to let $\delta_j = \delta < \min\{\frac{\varepsilon_i}{2(s-i)} \mid i = 1, \cdots, s-1\}$, $j = 1, \cdots, s$. Then the sequence $\{\mu_j\}$ will also be simplified significantly.

**Remark 4.** In searching for a common QLF, $P \sim kP$, $k > 0$, where " $\sim$ " stands for equivalence.

Next, we give a step-by-step algorithm for constructing a common QLF via cascading form.

**Algorithm 1.**

Step 1. For each $A_i$, find a set of basis under which $A_i$ can be expressed in Jordan canonical form.

Step 2. Compute all invariant subspaces of $A_i$.

1) Assume $A_i$ has eigenvalues $\lambda_1, \cdots, \lambda_\omega$. (For notational ease, without loss of generality we can assume $\omega = 2$.) Then, under basis $e_1^1, \cdots, e_1^s, e_2^1, \cdots, e_2^t$, $(s + t = n)$, $A_i$ can be expressed as

$$A_i = diag\{F_1(\lambda_1), F_2(\lambda_2)\}$$

where

$$F_1(\lambda_1) = diag\{J_{s_1}(\lambda_1), \cdots, J_{s_m}(\lambda_1)\}$$
$$F_2(\lambda_2) = diag\{J_{t_1}(\lambda_2), \cdots, J_{t_r}(\lambda_2)\}$$

and $J_k(\lambda_i)$ is in Jordan canonical form as in either (12) or (15), $s_1+, \cdots, +s_m = s$, $t_1 + \cdots, +t_r = t$, and

$$A_i[e_1^1, \cdots, e_1^s] = [e_1^1, \cdots, e_1^s]F_1(\lambda_1)$$
$$A_i[e_2^1, \cdots, e_2^t] = [e_2^1, \cdots, e_2^t]F_2(\lambda_2).$$

2) Denote $W_1 = Span\{e_1^1, \cdots, e_1^s\}$, $W_2 = Span\{e_2^1, \cdots, e_2^t\}$. Using Propositions 2 and 4, we can find a $\mu$-dimensional $A_i$-invariant subspace $U_\mu^1$ in $W_1$, $\mu = 1, \cdots, s$, and a $\nu$-dimensional $A_i$-invariant subspace $U_\nu^2$ in $W_2$, $\nu = 1, \cdots, t$.

3) Using Lemma 3, an $h$-dimensional $A_i$-invariant subspace can be obtained as $V_h^i = U_\mu^1 \oplus U_\nu^2$, where $\mu + \nu = h$.

Step 3. By looking for all possible $V_h^1 = V_h^2 = \cdots = V_h^N$, $h = 1, \cdots, n-1$, we can find whether there is an $h$-dimensional $A_i$-invariant $(i = 1, \cdots, N)$ subspace $V_h$.

Step 4. If we can find $0 \subsetneqq V_{h_1} \subsetneqq V_{h_2} \subsetneqq \cdots \subsetneqq V_{h_j} = \mathbb{R}^n$, to which all $A_i$ are invariant, where $V_{h_t} = Span\{e_1, \cdots, e_{h_t}\}$, then under the basis $T = [e_1, \cdots, e_{h_1}, e_{h_1+1}, \cdots, e_{h_2}, \cdots, e_n]$ $A_i$ can be simultaneously converted into cascading form (4), $i = 1, \cdots, N$.

Step 5. Using the method mentioned in [8] and Lemma 5, we may find a common QLF $Q$ of $T^{-1}A_iT$.

Step 6. With reference to the original coordinate frame, $(T^{-1})^{\mathrm{T}}Q(T^{-1})$ is a common QLF of $A_i$, $i = 1, \cdots, N$.

**Example 1.** Let

$$A_1 = \begin{pmatrix} 10 & -10.5 & 7 & -5 \\ \frac{10}{3} & -\frac{13}{3} & \frac{10}{3} & -2 \\ \frac{46}{3} & -\frac{43}{3} & \frac{19}{3} & -6 \\ 40 & -37.5 & 20 & -17 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -7 & 3 & -3 & 2 \\ 0 & -4 & 0 & 0 \\ -8 & 8 & -9 & 4 \\ -\frac{55}{3} & \frac{55}{3} & -\frac{40}{3} & 6 \end{pmatrix}.$$

Under basis $e_1 = (1, 0, 2, 5)^{\mathrm{T}}$, $e_2 = (3, 2, 3, 5)^{\mathrm{T}}$, $e_3 = (9, 8, 5, 10)^{\mathrm{T}}$, $e_4 = (5, 2, 6, 16)^{\mathrm{T}}$, $A_1$ can be converted into Jordan canonical form as

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Under basis $\eta_1 = (1, 0, 2, 5)^{\mathrm{T}}$, $\eta_2 = (-1, 0, 1, 0)^{\mathrm{T}}$, $\eta_3 = (2, 2, 0, 0)^{\mathrm{T}}$, $\eta_4 = (-2, 0, -4, -9)^{\mathrm{T}}$, $A_2$ can be converted into Jordan canonical form as

$$\begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

We will find common invariant sub-spaces of $A_1$ and $A_2$.

1) 1-dimensional invariant sub-space of $A_1$.

$$V_1^1 = Span\{a_1e_1 + b_1e_3\}$$
$$V_1^2 = Span\{e_4\}$$

1-dimensional invariant sub-space of $A_2$.

$$U_1^1 = Span\{c_1\eta_3 + d_1\eta_4\}$$
$$U_1^2 = Span\{\eta_1\}$$

2) 2-dimensional invariant sub-space of $A_1$

$$\begin{aligned}V_2^1 =& Span\{a_1e_1 + b_1e_3, a_2e_1 + b_2e_3\} = \\ & \text{(where } (a_1, b_1)\ (a_2, b_2) \text{ are linearly independent)} \\ & Span\{e_1, e_3\} \\ V_2^2 =& Z(e_2 + be_1 + ce_3, A_1 - (-1)I_4) = \\ & Span\{e_2 + be_1 + ce_3, e_1\} \\ V_2^3 =& V_1^1 \oplus V_1^2\end{aligned}$$

2-dimensional invariant sub-space of $A_2$

$$U_2^1 = Span\{\eta_1, \eta_2\}$$
$$U_2^2 = Span\{\eta_3, \eta_4\}$$
$$U_2^3 = U_1^1 \oplus U_1^2$$

3) 3-dimensional invariant sub-space of $A_1$.

$$V_3^1 = Span\{e_1, e_3, e_2\}$$
$$V_3^2 = V_2^1 \oplus V_1^2$$
$$V_3^3 = V_2^2 \oplus V_1^2$$

3-dimensional invariant sub-space of $A_2$

$$U_1^3 = U_2^1 \oplus U_1^1$$
$$U_2^3 = U_2^2 \oplus U_1^2$$

By solving a system of algebraic equations, we can find all common invariant spaces of $A_1$ and $A_2$. Precisely, since $U_2^3 = V_2^1$, if we choose $a_1 = 1, b_1 = 0$, $a_2 = -1, b_2 = 1$, and $c_1 = 1, d_1 = 0$, then under $T = [\eta_1, \eta_3, \eta_2, \eta_4]$, $A_1$ and $A_2$ have the same block upper triangle structure as

$$T^{-1}A_1T = \begin{pmatrix} -1 & 1 & -4 & -5 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$
$$T^{-1}A_2T = \begin{pmatrix} -3 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

According to Theorem 1, if

$$A_1^{11} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, A_2^{11} = \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}$$

have a common QLF, and

$$A_1^{22} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, A_2^{22} = \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}$$

have a common QLF, then so do $A_1$ and $A_2$. Using the method proposed in [8], we can find common QLF $P_1$ and $P_2$ as

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}.$$

It is easy to check that

$$P_1A_1^{11} + (A_1^{11})^{\mathrm{T}}P_1 = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix} < 0$$
$$P_1A_2^{11} + (A_1^{11})^{\mathrm{T}}P_1 = \begin{pmatrix} -6 & 0 \\ 0 & -16 \end{pmatrix} < 0$$
$$P_2A_1^{22} + (A_1^{22})^{\mathrm{T}}P_2 = \begin{pmatrix} -6 & 0 \\ 0 & -16 \end{pmatrix} < 0$$
$$P_2A_2^{22} + (A_2^{22})^{\mathrm{T}}P_2 = \begin{pmatrix} -18 & 0 \\ 0 & -32 \end{pmatrix} < 0.$$

According to Lemma 5, for a large enough $\mu_2 > 0$, $Q = diag\{P_1, \mu P_2\}$ is a common QLF of $T^{-1}A_1T$ and $T^{-1}A_2T$. With a direct computation we can choose $\delta_1 = 1.5$, $\varepsilon_2 = 5$, and $M_2 = 44$, so $\mu_2 > 8$ is enough. Choosing $\mu_2$=10, and referring back to the original coordinate frame, we have

$$P = (T^{-1})^{\mathrm{T}}Q(T^{-1}) = \begin{pmatrix} 133.4444 & -133.4444 & 113.4444 & -72.6667 \\ -133.4444 & 133.9444 & -113.4444 & 72.6667 \\ 113.4444 & -113.4444 & 123.4444 & -72.6667 \\ -72.6667 & 72.6667 & -72.6667 & 44.0000 \end{pmatrix}.$$

Then we can check that

$$PA_1 + A_1^{\mathrm{T}}P = \begin{pmatrix} -555.1098 & 555.2764 & -503.1108 & 315.6667 \\ 555.2764 & -556.4431 & 503.2774 & -315.6667 \\ -503.1108 & 503.2774 & -511.1118 & 307.6667 \\ 315.6667 & -315.6667 & 307.6667 & -188.0000 \end{pmatrix} < 0$$

$$PA_2 + A_2^{\mathrm{T}}P = 10^3 * \begin{pmatrix} -1.0189 & 1.0189 & -0.9019 & 0.5680 \\ 1.0189 & -1.0229 & 0.9019 & -0.5680 \\ -0.9019 & 0.9019 & -0.9649 & 0.5700 \\ 0.5680 & -0.5680 & 0.5700 & -0.3440 \end{pmatrix} < 0.$$

Hence $P$ is a common QLF of $A_1$ and $A_2$. □

## 5 Quadratic stabilization

In this section, we consider the problem of the feedback quadratic stabilization of a switched linear system (3). Consider system (6), for a given sequence of nested subspaces of $\mathbb{R}^n$,

$$0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_{s-1} \subsetneqq V_s$$

We look for feedback $u = Kx$, such that a sequence of nested subspaces become the quasi-flag of the feedback closed-loop of (6). That is, there exists $K$ such that

$$(A + BK)V_i \in V_i, \quad i = 1, \cdots, s. \tag{23}$$

In the following proposition, we will construct the matrix $K$.

**Proposition 7.** (Cascading Quaker Lemma) For a given sequence of nested subspaces of $\mathbb{R}^n$

$$0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_{s-1} \subsetneqq V_s = \mathbb{R}^n$$

there exits feedback $K$ such that (23) holds, iff $V_i$ is $(A, B) - invariant$, $i = 1, \cdots, s-1$.

From the previously described coordinate frame, we can find $K$ such that $A + BK$ has block upper triangular form

$$A + BK = \begin{pmatrix} A^{11} & \times & \cdots & & \times \\ \times & A^{22} & \cdots & & \times \\ \vdots & & & & \\ \times & \times & \cdots & \times & A^{ss} \end{pmatrix} + \begin{pmatrix} B^1 \\ B^2 \\ \vdots \\ B^s \end{pmatrix} \begin{pmatrix} K^1 & K^2 & \cdots & K^s \end{pmatrix} = \begin{pmatrix} \tilde{A}^{11} & \times & \cdots & & \times \\ 0 & \tilde{A}^{22} & \cdots & & \times \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & \tilde{A}^{ss} \end{pmatrix}. \tag{24}$$

Then we have the following

**Proposition 8.** Assume there exists $K$ such that (24) holds. Moreover, $(\tilde{A}^{ii}, B^i)$, $i = 1, \cdots, s$ are stabilizable. Then system (6) is stabilizable.

Using the above approach to a switched linear system (3), we have the following.

**Proposition 9.** Assume that there exists a quasi-flag and with respect to this flag all switched models are $(A_k, B_k) - invariant$ such that $A_k + B_k K_k$, $k \in \Lambda$ have the form (24). Then system (3) is quadratically stabilizable if

$$\dot{z} = A_k^{ii} z + B_k^i u_i, \quad k \in \Lambda \quad i = 1, \cdots, s$$

are quadratically stabilizable.

## 6 Conclusion

This paper provides a systematic method to find all possible real quasi-flags for a given matrix. With it, all block upper triangular forms are obtained. It follows that all comparable upper triangular forms for a set of matrices can also be obtained. When the Lie algebra generated by a set of matrices is reducible, a switched linear system can always be expressed in cascading form. Using this form the verification of quadratic stability for a switched system can be simplified. A detailed algorithm is provided for this procedure of cascading transformation. Finally, a cascading Quaker Lemma is proven, and by combining it with stability results a quadratic stabilization problem was investigated.

## Appendix

**Proof of Proposition 1.** (Necessity) Assume $A_i$ has the form (4). Let $V_j = \text{Span col}\left\{\begin{pmatrix} I_{d_j} \\ 0 \end{pmatrix}\right\}$, $j = 1, \cdots, s$, where $d_j = n_1 + \cdots + n_j$. Then it is easy to see that for each $V_t \in \{V_j\}$, $V_t$ is $A_i$ invariant.

(Sufficiency) Construct an $n \times n$ matrix $T$ as

$$T = \begin{pmatrix} \xi_1^1 & \cdots & \xi_{n_1}^1 & \xi_1^2 & \cdots & \xi_{n_2}^2 & \cdots & \xi_1^s & \cdots & \xi_{n_s}^s \end{pmatrix}$$

where the set $\{\xi_1^1, \cdots, \xi_{n_1}^1, \cdots, \xi_1^j, \cdots, \xi_{n_j}^j\}$ is the basis of $V_j$, $j = 1, \cdots, s$. Then it is easy to check that $T^{-1}A_iT$ has the form (4). □

**Proof of Proposition 2.** Note that

$$H \supset \sum_{j=1}^{\tau} Z(\xi_j, C)$$

and $\dim(H) = t$, $\dim(Z(\xi_j, C)) = k_j$, and $\sum_{j=1}^{\tau} k_j = t$. So to prove $H = \oplus_{j=1}^{\tau} Z(\xi_j, C)$, it suffices to show that all $\{Z(\xi_j, C) | 1 \leqslant j \leqslant \tau\}$ are linearly independent. Therefore, it is enough to show that

$$C_u := Z(\xi_u, C) \cap \left( \sum_{v=\{1,\cdots,\tau\}\backslash u} Z(\xi_v, C) \right) = \{0\}$$
$$u = 1, \cdots, \tau. \tag{A1}$$

We prove (A1) by contradiction. Assume $C_u \neq \{0\}$. A simple computation shows that

$$AT = T \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}_{k_j \times k_j}$$

where $T = [C^{k_j-1}\xi_j, C^{k_j-2}\xi_j, \cdots, C\xi_j, \xi_j]$. It is obvious that the restriction of $A$ to $Z(\xi_j, C)$, denoted by $A|_{Z(\xi_j,C)}$, has only one eigenvector $C^{k_j-1}\xi_j = \sum_{i=1}^{\tau} c_{i,k_j}^j e_i^1$, $j = 1, \cdots, \tau$. Meanwhile, the eigenvector set of $\sum_{v=\{1,\cdots,\tau\}\backslash u} Z(\xi_v, C)$ is

$$D := span\{C^{k_1-1}\xi_1, \cdots, C^{k_{u-1}-1}\xi_{u-1}, C^{k_{u+1}-1}\xi_{u+1}, \cdots, C^{k_\tau-1}\xi_\tau\}$$

Because $C_u \neq \{0\}$, $C^{k_u-1}\xi_u \in D$. Therefore, we have the following equation

$$\sum_{v=\{1,\cdots,\tau\}\backslash u} a_v C^{k_v-1}\xi_v = C^{k_u-1}\xi_u. \tag{A2}$$

Denote

$$E_u = \begin{pmatrix} c^1_{1,k_1}, & \cdots, & c^{u-1}_{1,k_{u-1}} & c^{u+1}_{1,k_{u+1}}, & \cdots, & c^{\tau}_{1,k_\tau} \\ \vdots & & \vdots & \vdots & & \\ c^1_{m,k_1}, & \cdots, & c^{u-1}_{m,k_{u-1}} & c^{u+1}_{m,k_{u+1}}, & \cdots, & c^{\tau}_{m,k_\tau} \end{pmatrix}_{m\times(\tau-1)}.$$

Denoting $X_u = (a_1, \cdots, a_{u-1}, a_{u+1}, \cdots, a_\tau)^{\rm T}$, $Y_u = (c_1^{k_u}, \cdots, c_m^{k_u})^{\rm T}$, (A2) becomes

$$E_u X_u = Y_u.$$

Because $(c^j_{1,k_j}, \cdots, c^j_{m,k_j})^{\rm T}, j = 1, \cdots, \tau$ are linearly independent, $rank(E_u|Y_u) \neq rank(E_u)$. Hence (A2) has no solution. It follows that $C^{k_u-1}\xi_u$ doesn't belong to $D$, which leads to a contradiction. So $C_u = \{0\}$. This ends the proof. □

**Proof of Proposition 3.** Using Lemma 4, $H$ can be decomposed as the direct sum of several (we say $\tau$) $A$-cyclic subspaces, that is, $H = \oplus_{j=1}^{\tau} Z(\xi_j, C)$, and $\dim(Z(\xi_j, C)) = k_j$, $j = 1, \cdots, \tau$. Since $\xi_j \in \mathbb{R}^n$, $\xi_j$ can be expressed as $\xi_j = \sum\limits_{i=1}^{m}\sum\limits_{l=1}^{s_i} c^j_{i,l} e^l_i$, with a direct computation we know

$$\begin{aligned}&\dim(Z(\xi_j, C)) = max\{\ l \mid c^j_{i,l} \neq 0,\\ &i = 1, \cdots, m, \quad l = 1, \cdots, s_i\} = k_j.\end{aligned} \tag{A3}$$

So $c^j_{i,l} = 0$, $l = k_j+1, \cdots, s_j$, $i = 1, \cdots, m$, then we establish that $\xi_j$ has the form (13). Since $\{Z(\xi_j, C)|1 \leqslant j \leqslant \tau\}$ are linearly independent, that is, $C_u = 0$, by the proof of Proposition 2 we know $E_u X_u = Y_u$ has no solution, and $[c^j_{1,k_j}, c^j_{2,k_j}, \cdots, c^j_{m,k_j}]^{\rm T}$, $j = 1, \ldots, \tau$ are linearly independent. This ends the proof. □

**Proof of Proposition 4.** Similar to the proof of Proposition 2, the dimension estimate shows that to prove

$$H = \oplus_{j=1}^{\tau} Z(\xi_j, A)$$

we have only to show that $\{\, Z(\xi_j, A)|\, j = 1, \cdots, \tau\}$ are linearly independent. So it suffices to show that

$$\begin{aligned}C_u :=& Z(\xi_u, A) \cap \left(\sum_{v=\{1,\cdots,\tau\}\backslash u} Z_{k_v}(\xi_v, A)\right) = \\ &\{0\}, \quad u = 1, \cdots, \tau.\end{aligned} \tag{A4}$$

We prove (A4) by contradiction. Assume $C_u \neq \{0\}$. A straightforward computation shows that

$$AT = T\begin{pmatrix} \alpha & \beta & 0 & 0 & ,\cdots, & 0 & 0 \\ -\beta & \alpha & 1 & 0 & ,\cdots, & 0 & 0 \\ 0 & 0 & \alpha & \beta & ,\cdots, & 0 & 0 \\ 0 & 0 & -\beta & \alpha & ,\cdots, & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & ,\cdots, & 0 & 0 \\ 0 & 0 & 0 & 0 & ,\cdots, & 1 & 0 \\ 0 & 0 & 0 & 0 & ,\cdots, & \alpha & \beta \\ 0 & 0 & 0 & 0 & ,\cdots, & -\beta & \alpha \end{pmatrix}_{2k_j\times 2k_j} \tag{A5}$$

where

$$\begin{aligned}T =& [\frac{B^{k_j-1}(A-\alpha I)}{\beta^{k_j}}\xi_j, \frac{B^{k_j-1}}{\beta^{k_j-1}}\xi_j, \cdots, \frac{B^2}{\beta^2}\xi_j,\\ &\frac{B(A-\alpha I)}{\beta^2}\xi_j, \frac{B}{\beta}\xi_j, \frac{(A-\alpha I)}{\beta}\xi_j, \xi_j].\end{aligned}$$

Denote $Q_j$ as the restriction of $A$ to $Z(A, \xi_j)$, i.e. $Q_j = A|_{Z(A,\xi_j)}$, $j = 1, \cdots, \tau$. From (A5) we know that $ker\left(Q_j^2 - 2\alpha Q_j + (\alpha^2+\beta^2)I\right)$ is spanned by a set of linearly independent real vectors, and one of these sets is

$$\delta_{j1} = \frac{B^{k_j-1}}{\beta^{k_j-1}}\xi_j = \sum_{i=1}^{m}\left(c^j_{i,2k_j-1}e^1_i + c^j_{i,2k_j}e^2_i\right)$$

$$\delta_{j2} = \frac{B^{k_j-1}(A-\alpha I)}{\beta^{k_j}}\xi_j = \sum_{i=1}^{m}(c^j_{i,2k_j}e^1_i - c^j_{i,2k_j-1}e^2_i)$$

Meanwhile, if we denote $\bar{Q}_u = A|_{\sum\limits_{v=\{1,\cdots,\tau\}\backslash u} Z(\xi_v, A)}$, then the set of $ker\left(\bar{Q}_u^2 - 2\alpha\bar{Q}_u + (\alpha^2+\beta^2)I\right)$ is

$$D := span\{\delta_{v1}, \delta_{v2} \quad | \quad v = 1, \cdots, u-1, u+1, \cdots, \tau\}.$$

Because $C_u \neq \{0\}$, it is easy to verify that $C_u$ is $A$-invariant, then $dim(C_u) = 2$, so $\delta_{u1} \in D$ and $\delta_{u2} \in D$. Hence, we have the following two equations:

$$\sum_{v=\{1,\cdots,\tau\}\backslash u} a_v\delta_{v1} + b_v\delta_{v2} = \delta_{u1} \tag{A6}$$

$$\sum_{v=\{1,\cdots,\tau\}\backslash u} c_v\delta_{v1} + d_v\delta_{v2} = \delta_{u2}. \tag{A7}$$

Denote (A8) (A9).

Define

$$\begin{aligned}X_{u1} &= (a_1, \cdots, a_{u-1}, a_{u+1}, \cdots, a_\tau)^{\rm T}\\ X_{u2} &= (b_1, \cdots, b_{u-1}, b_{u+1}, \cdots, b_\tau)^{\rm T}\\ Y_{u1} &= (c^u_{1,2k_u-1}, \cdots, c^u_{m,2k_u-1})^{\rm T}\\ Y_{u2} &= (c^u_{1,2k_u}, \cdots, c^u_{m,2k_u})^{\rm T}.\end{aligned}$$

Then (A6) becomes

$$\begin{pmatrix} E_{u1} & E_{u2} \\ E_{u2} & -E_{u1} \end{pmatrix} \begin{pmatrix} X_{u1} \\ X_{u2} \end{pmatrix} = \begin{pmatrix} Y_{u1} \\ Y_{u2} \end{pmatrix}.$$

Since $[c^j_{1,2k_j-1}, \cdots, c^j_{m,2k_j-1}, c^j_{1,2k_j}, \cdots, c^j_{m,2k_j}]$, $[c^j_{1,2k_j}, \cdots, c^j_{m,2k_j}, -c^j_{1,2k_j-1}, \cdots, -c^j_{m,2k_j-1}]$, $j = 1, \cdots, \tau$, are linearly independent, (A6) has no solution. Similarly, (A7) has no solution. This leads to a contradiction. So $C_u = \{0\}$, and hence $H = \oplus_{j=1}^{\tau} Z(\xi_j, A)$. □

**Proof of Proposition 6.** Note that the product of the same structures of upper triangular matrixes makes the structure unchanged, as with the Lie bracket $[A, B] = AB - BA$. □

**Proof of Lemma 5.** Denote $H_i = PA_i + A_i^{\mathrm{T}} P$; then (A10).

To show $H_i < 0$, choose $\xi_k \in \mathbb{R}^{n_k}$, $k = 1, \cdots, s$. Then

$$(\xi_1^{\mathrm{T}}, \xi_2^{\mathrm{T}}, \cdots, \xi_s^{\mathrm{T}}) H_i \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_s \end{pmatrix} =$$
$$-\sum_{k=1}^{s} \mu_k \xi_k^{\mathrm{T}} V_i^k \xi_k + \sum_{j=1}^{s-1} \sum_{k=j+1}^{s} \mu_j \cdot$$
$$[\xi_k^{\mathrm{T}} (A_i^{jk})^{\mathrm{T}} P_j \xi_j + \xi_j^{\mathrm{T}} P_j A_i^{jk} \xi_k] \leqslant$$
$$-\sum_{k=1}^{s} \mu_k \xi_k^{\mathrm{T}} V_i^k \xi_k + \sum_{j=1}^{s-1} \sum_{k=j+1}^{s} \mu_j [\delta_j \|\xi_j\|^2 +$$
$$\frac{1}{\delta_j} \xi_k^{\mathrm{T}} (A_i^{jk})^{\mathrm{T}} P_j^2 A_i^{jk} \xi_k] =$$
$$\xi_1^{\mathrm{T}} (-\mu_1 V_i^1 + (s-1)\delta_1 I_{n_1}) \xi_1 +$$
$$\sum_{k=2}^{s} \xi_k^{\mathrm{T}} [-\mu_k (V_i^k - (s-k)\delta_k I_{n_k}) +$$
$$\sum_{j=1}^{k-1} \mu_j (\frac{1}{\delta_j} (A_i^{jk})^{\mathrm{T}} P_j^2 A_i^{jk})] \xi_k. \tag{A11}$$

Then it is easy to check that when (20) is satisfied, $-\mu_1 V_i^1 + (s-1)\delta_1 I_{n_1}$ is a symmetric negative definite matrix, and so are the matrixes $-\mu_k(V_i^k - (s-k)\delta_k I_{n_k}) + \sum_{j=1}^{k-1} \mu_j (\frac{1}{\delta_j}(A_i^{jk})^{\mathrm{T}} P_j^2 A_i^{jk})$, $k = 2, \cdots, s$. It follows that $H_i < 0$.

**Proof of Proposition 7.** Necessity is obvious, we prove sufficiency. Suppose $V_1 = Span\{e_1, \cdots, e_{n_1}\}$, $V_2 = V_1 \cup Span\{e_{n_1+1}, \cdots, e_{n_2}\}$, $\cdots$, $V_{s-1} = V_{s-2} \cup Span\{e_{n_{s-2}+1}, \cdots, e_{n_{s-1}}\}$, $V_s = V_{s-1} \cup Span\{e_{n_{s-1}+1}, \cdots, e_{n_s}\}$. Because $V_i$ is $(A, B) - invariant$,

$$Ae_1 = \omega_1 - B\mu_1, \quad \text{where} \quad \omega_1 \in V_1$$
$$\vdots$$
$$Ae_{n_1} = \omega_{n_1} - B\mu_{n_1} \quad \text{where} \quad \omega_{n_1} \in V_1$$
$$Ae_{n_1+1} = \omega_{n_1+1} - B\mu_{n_1+1} \quad \text{where} \quad \omega_{n_1+1} \in V_2$$
$$\vdots$$
$$Ae_{n_{s-1}} = \omega_{n_{s-1}} - B\mu_{n_{s-1}} \quad \text{where} \quad \omega_{n_{s-1}} \in V_{s-1}.$$

Setting $K = (\mu_1, \mu_2, \cdots, \mu_{n_{s-1}}, \mu_{n_{s-1}+1}, \cdots, \mu_{n_s})(e_1, \cdots, e_{n_s})^{-1}$, where $\mu_{n_{s-1}+1}, \mu_{n_{s-1}+2}, \cdots, \mu_{n_s}$ are arbitrary $m - dimensional$ vectors, it is easy to check that $(A + BK)V_i \in V_i$, $i = 1, \cdots, s$. □

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$$E_{u1} = \begin{pmatrix} c^1_{1,2k_1-1}, & \cdots, & c^{u-1}_{1,2k_{u-1}-1} & c^{u+1}_{1,2k_{u+1}-1}, & \cdots, & c^{\tau}_{1,2k_\tau-1} \\ \vdots & & \vdots & \vdots & & \\ c^1_{m,2k_1-1}, & \cdots, & c^{u-1}_{m,2k_{u-1}-1} & c^{u+1}_{m,2k_{u+1}-1}, & \cdots, & c^{\tau}_{m,2k_\tau-1} \end{pmatrix}_{m\times(\tau-1)} \tag{A8}$$

$$E_{u2} = \begin{pmatrix} c^1_{1,2k_1}, & \cdots, & c^{u-1}_{1,2k_{u-1}} & c^{u+1}_{1,2k_{u+1}}, & \cdots, & c^{\tau}_{1,2k_\tau} \\ \vdots & & \vdots & \vdots & & \\ c^1_{m,2k_1}, & \cdots, & c^{u-1}_{m,2k_{u-1}} & c^{u+1}_{m,2k_{u+1}}, & \cdots, & c^{\tau}_{m,2k_\tau} \end{pmatrix}_{m\times(\tau-1)} \tag{A9}$$

$$H_i = \begin{pmatrix} -\mu_1 V_i^1 & \mu_1 P_1 A_i^{12} & \mu_1 P_1 A_i^{13} & \cdots & \mu_1 P_1 A_i^{1(s-1)} & \mu_1 P_1 A_i^{1s} \\ \mu_1 (A_i^{12})^{\mathrm{T}} P_1 & -\mu_2 V_i^2 & \mu_2 P_2 A_i^{23} & \cdots & \mu_2 P_2 A_i^{2(s-1)} & \mu_2 P_2 A_i^{2s} \\ \mu_1 (A_i^{13})^{\mathrm{T}} P_1 & \mu_2 (A_i^{23})^{\mathrm{T}} P_2 & -\mu_3 V_i^2 & \cdots & \mu_3 P_3 A_i^{3(s-1)} & \mu_3 P_3 A_i^{3s} \\ \vdots & & & & & \\ \mu_1 (A_i^{1s})^{\mathrm{T}} P_1 & \mu_2 (A_i^{2s})^{\mathrm{T}} P_2 & \mu_3 (A_i^{3s})^{\mathrm{T}} P_3 & \cdots & \mu_{s-1} (A_i^{(s-1)s})^{\mathrm{T}} P_{s-1} & -\mu_s V_i^s \end{pmatrix} \tag{A10}$$

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**Ya-Hong Zhu** received her B.Sc. degree in Department of Mathematics from Soochow University, Suzhou, China in 2002. Since 2002, she has been a postgraduate student in the Institute of Systems Science, Chinese Academy of Sciences.

Her research interests include nonlinear control systems, switched systems.

**Dai-Zhan Cheng** graduated from Tsinghua University in 1970, received his M.Sc from Graduate School, Chinese Academy of Sciences (CAS) in 1981, Ph.D. from Washington University in 1985. Since 1990, he is a professor with the Institute of Systems Science, Academy of Mathematics and Systems Science (AMSS), CAS.

He is the author/coauthor of 160 journal papers, 6 books and 80 conference papers. His research interests include nonlinear control systems, switched systems, Hamiltonian systems and numerical realization for control design.

Prof. Cheng is currently chairman of Technical Committee on Control Theory, CSA, and IEEE Fellow.