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Systems & Control Letters 55 (2006) 558–565
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Optimal impulsive control in periodic ecosystem

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Received 21 April 2003; received in revised form 28 June 2005; accepted 14 December 2005
Available online 20 March 2006

Abstract

In this paper, the impulsive exploitation of single species modelled by periodic Logistic equation is considered. First, it is shown that the generally periodic Kolmogorov system with impulsive harvest has a unique positive solution which is globally asymptotically stable for the positive solution. Further, choosing the maximum annual biomass yield as the management objective, we investigate the optimal harvesting policies for periodic logistic equation with impulsive harvest. When the optimal harvesting effort maximizes the annual biomass yield, the corresponding optimal population level, and the maximum annual biomass yield are obtained. Their explicit expressions are obtained in terms of the intrinsic growth rate, the carrying capacity, and the impulsive moments. In particular, it is proved that the maximum biomass yield is in fact the maximum sustainable yield (MSY). The results extend and generalize the classical results of Clark [Mathematical Bioeconomics: The Optimal Management of Renewable Resources, Wiley, New York, 1976] and Fan [Optimal harvesting policy for single population with periodic coefficients, Math. Biosci. 152 (1998) 165–177] for a population described by autonomous or nonautonomous logistic model with continuous harvest in renewable resources.

Keywords: Optimal impulsive harvesting policy; Logistic equation; Impulsive effects; Periodic solutions; Maximum annual-sustainable yield

1. Introduction

The optimal management of renewable resources, which has a direct relationship to sustainable development, has received much attention for a long time [8,1,11]. In the simplest sense, sustainable use of a resources means that the resource can be used indefinitely. Moreover, one always hopes to achieve sustainability at a high level of productivity. Fish resources are very important renewable resources. Sustainability of fisheries at a good level of productivity and of the economic results requires a relatively broad understanding of appropriate and effective management. In practice, the management of fishing is a decision with multiple objectives [11]. Some of the desirable objectives in the management of fish resources are as follows: (1) the provision of good biomass yield, (2) the conservation of fish population, (3) the provision of good economic returns, (4) the conservation of genetic variability of the fish population, and so on. The formulation of good harvesting policies which take into account these objectives is a complex and difficult task even if the dynamics of a fish population is known accurately and the objectives are fully quantified. One practical way to overcome the mathematical difficulties in this complex problem is to convert tentatively all the objective functions except one into constraints and to optimize the only remaining objective function.

Managing single-species fisheries with above objectives has been studied systemically. Suppose that $x(t)$, the density of the fish population at time t, satisfies the well known logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \tag{1.1}$$

where r, as a positive constant, is called the intrinsic growth rate, the positive constant K is usually referred to as the environment carrying capacity, or saturation level. Now,

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1 Supported by the National Natural Science Foundation of China.

0167-6911/$ - see front matter
doi:10.1016/j.sysconle.2005.12.003

suppose that the resource population described by the logistic Eq. (1.1) is subject to exploitation, under the catch-per-unit-effort hypothesis

$$h = Ex.$$

The equation of the harvested population reads

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - Ex, \tag{1.2}$$

where the positive constant E denotes the harvesting effort. In order to gain the maximum biomass yield, Clark [8] considered this optimal harvesting problem, and obtained the optimal harvesting effort, the corresponding optimal population level, respectively. Later Fan et al. [10] considered the optimal harvest problem for the modified model (1.2) with periodic coefficients

$$\dot{x} = r(t)x\left(1 - \frac{x}{K(t)}\right) - E(t)x, \tag{1.3}$$

and generalized the classical results of Clark.

A common assumption for both model (1.2) and model (1.3) is that the human exploit activities occur continuously. However, this is not how the thing looks like. Mostly, the harvesting of species is seasonal or it occurs in regular pulses. The continuous human's exploit action is then removed from the model, and replaced by an impulsive perturbation. These models are subject to short-term perturbations which are often assumed to be in the form of impulses in the modelling process. Consequently, the impulsive differential equations provide a natural description of such systems [4,5,16]. Equations of this kind appear in almost every domain of applied sciences. Some numerous examples were given in Bainov's and his collaborators' books [4,5]. Some impulsive equations have been recently introduced in population dynamics which are related with population ecology [17,6,2,9,21,22] and chemotherapeutic treatment of disease [15,19], etc.

The purpose of this paper is to investigate the effect of impulsive perturbation on the growth of population and further to study what impulsive harvesting policies can be used to maximize the biomass yield and to make the population sustainable development. The rest of the paper is organized as follows: in the following section, we first discuss the general nonlinear differential equation with impulsive harvest and show that there exists a unique periodic solution which globally attracts all other positive solutions, that is, system is permanent (see definition in [6]). Section 3 focuses on the optimal harvesting problem for periodic logistic equation with impulsive effect. The optimal harvesting effort that maximizes the biomass yield, and the corresponding optimal population level are determined. In particular, it is shown that the optimal population level is truly the positive periodic solution of the considered system, and hence it is globally asymptotically stable. Moreover, the corresponding maximum biomass yield is the maximum sustainable yield (MSY). In Section 4, we compare the optimal impulsive harvesting policy with optimal continuous harvesting policy by using both theoretical analysis and numerical method, which demonstrate the advantage of optimal impulsive harvesting policy.

2. Existence and uniqueness of periodic solution

Let $\tau_1 < \tau_2 < \cdots$ be sequence of positive numbers. We define $PC = \{\psi : R \to R, \psi$ is continuous for $t \neq \tau_k$, $\psi(\tau_k^+)$ and $\psi(\tau_k^-)$ exist and $\psi(\tau_k) = \psi(\tau_k^-)$, $k = 1, 2, \ldots\}$; $PC' = \{\psi \in PC :$ ψ is differentiable at $t \neq \tau_k$, $k = 1, 2, \ldots\}$, where $\psi(\tau_k^+) = \lim_{h\to 0^+} \psi(\tau_k + h)$. Denote PC_T (respectively, $PC'_T) = \{\psi \in PC$ (respectively, PC') $: \psi(t+T) = \psi(t), t \in R\}$.

Consider the following Kolmogorov-type equation with impulsive harvest:

$$\dot{x}(t) = x(t)F(t, x(t)), \quad t \neq \tau_k, \quad k \in N, \tag{2.1}$$

$$x(\tau_k^+) = x(\tau_k)(1 - E_k), \quad t = \tau_k, \quad k \in N, \tag{2.2}$$

where $F : R \times R \to R$ is a continuous function and $N = \{1, 2, \ldots\}$. From the point view of biology, we assume $0 < E_k < 1$, $k = 1, 2, \ldots$. The systems (2.1) and (2.2) are said to be periodic with period $T > 0$ if there exists a positive integer q such that

$$F(t+T, x) = F(t, x), \quad E_{k+q} = E_k, \quad \tau_{k+q} = \tau_k + T. \tag{2.3}$$

Further, we assume that F is locally Lipschitz with respect to the second variable. Then we can prove the existence and uniqueness of the solution of systems (2.1) and (2.2) with a given initial value.

Theorem 2.1. *Suppose systems* (2.1) *and* (2.2) *satisfy* (2.3). *In addition*,

(a) $F(t, x) > F(t, y)$ *if* $0 \leqslant x < y$;
(b) *there exists a positive constant* M *such that* $F(t, M) \leqslant 0$ *for all* t;
(c) $[\prod_{k=1}^{q}(1 - E_k)]\mathrm{e}^{\int_0^T F(t,0)\,\mathrm{d}t} > 1$.

Then systems (2.1) and (2.2) have exactly a T-periodic and positive solution $x^p(t) \in PC'_T$, with $x^p(t) \leqslant M$. Moreover, the solution $u(t) \triangleq u(t; 0, x_0)$ of Eqs. (2.1) and (2.2) with the initial value $x_0 > 0$ at $t = 0$ is defined on $[0, +\infty)$ and

$$\lim_{t\to+\infty} |u(t) - x^p(t)| = 0. \tag{2.4}$$

Proof. Let $u(t) \in PC'$ be a solution of (2.1) and (2.2), then $u(t) > 0$ for all t in the domain $\mathrm{Dom}(u)$ of u. On the other hand, it is easy to prove that

$$u(t) \leqslant \max\{u(0), M\}, \quad t \in [0, +\infty) \cap \mathrm{Dom}(u). \tag{2.5}$$

Thus, $u(t)$ is bounded on $t \in [0, +\infty) \cap \mathrm{Dom}(u)$ and, hence, $u(t)$ is defined in $[0, +\infty)$.

For $x \geqslant 0$, let $u(t, x)$ be the solution of the systems (2.1) and (2.2) such that $u(0, x) = x$. Define the Poincaré map (for properties of the Poincaré map of impulsive differential equation, see for instance [12,20]): $P:[0,+\infty)\to[0,+\infty)$ by $P(x)=u(T,x)$. Then we have $P(0)=0$, $\dot{P}(0)=[\prod_{k=1}^{q}(1-E_k)]\mathrm{e}^{\int_0^T F(t,0)\,\mathrm{d}t}>1$, and by (2.2), $P(M)\leqslant M$. Hence, P has at least one fixed point $z\in(0,M]$ such that $u(T,z)=z$, which implies that $x^p(t):=u(t,z)$ is a T-periodic and positive solution of the systems (2.1) and (2.2).

Let V, W be positive and T-periodic solutions of the systems (2.1) and (2.2) and let $h=V/W-1$. Then, h is T-periodic and

$$\dot{h}(t)=\frac{V(t)}{W(t)}[F(t,V(t))-F(t,W(t))],\quad t\neq\tau_k,\ \ k\in N,$$

$$h(\tau_k^+)=\frac{V(\tau_k)}{W(\tau_k)}-1=h(\tau_k),\quad t=\tau_k,\ \ k\in N. \tag{2.6}$$

If $V(t)\neq W(t)$, without loss of generality, we can assume that $V(t)<W(t)$ due to the uniqueness of solutions of systems (2.1) and (2.2). By condition (a) of Theorem 2.1 and (2.6), we get $\dot{h}(t)>0, t\neq\tau_k$ and $h(\tau_k^+)=h(\tau_k)$, which implies that h cannot be periodic and this contraction proves that systems (2.1) and (2.2) have exactly a T-periodic and positive solution. Equivalently, z is the unique fixed point of P in $(0,+\infty)$.

Let $u(t)=u(t,x_0)$ be a solution of (2.1) and (2.2). It is easy to see that $u(t+T)$ is also a solution of (2.1) and (2.2). In fact, let $y(t)=u(t+T)$, we have for all $k\in N$,

$$\dot{y}(t)=\dot{u}(t+T)=u(t+T)F(t+T,u(t+T))$$
$$=y(t)F(t,y(t)),\quad t\neq\tau_k,$$

$$y(\tau_k^+)=u(\tau_k^++T)=u(\tau_{k+q}^+)=u(\tau_{k+q})(1-E_{k+q})$$
$$=y(\tau_k)(1-E_k),\quad t=\tau_k,$$

which shows that $u(t+T)$ is also a solution of (2.1) and (2.2), and we can see that $\{u(nT),n=1,2,\ldots\}$ is a positive monotonic sequence of R. In particular, $u(nT)\to u_0$ as $n\to+\infty$, for some $u_0\geqslant 0$.

We claim that $u_0>0$. Otherwise, assume $u_0=0$, then $\{u(nT)\}$ is strictly decreasing and $u(t)\to 0$ as $t\to+\infty$. Now, let us choose $\varepsilon>0$ sufficiently small such that $[\prod_{k=1}^{q}(1-E_k)]\mathrm{e}^{\int_0^T F(t,\varepsilon)\,\mathrm{d}t}>1$ and there exists $t_1>t_0$ such that $u(t)\leqslant\varepsilon$ for all $t\geqslant t_1$. Integrating Eq. (2.1) from nT to $nT+\tau_1$, $nT+\tau_{k-1}$ to $nT+\tau_k$ $(k=2,\ldots,q)$ and from $nT+\tau_q$ to $nT+T$ we get

$$\ln\frac{u(nT+\tau_1)}{u(nT)}=\int_{nT}^{nT+\tau_1}F(t,u(t))\,\mathrm{d}t,$$
$$\ln\frac{u(nT+\tau_2)}{u(nT+\tau_1^+)}=\int_{nT+\tau_1}^{nT+\tau_2}F(t,u(t))\,\mathrm{d}t,$$
$$\vdots$$
$$\ln\frac{u(nT+\tau_{q+1})}{u(nT+\tau_q^+)}=\int_{nT+\tau_q}^{nT+T}F(t,u(t))\,\mathrm{d}t.$$

Then we have

$$0>\ln\frac{u(nT+T)}{u(nT)}=\ln\left[\prod_{k=1}^{q}(1-E_k)\right]+\int_{nT}^{nT+T}F(t,u(t))\,\mathrm{d}t\geqslant\ln\left\{\left[\prod_{k=1}^{q}(1-E_k)\right]\mathrm{e}^{\int_0^T F(t,\varepsilon)\,\mathrm{d}t}\right\}>0$$

for all integers $n\geqslant t_1$. This contradiction proves the Claim.

From this, u_0 is a positive fixed point of P since $P^n(x_0)=u(nT)\to u_0$. Consequently, $u_0=z$ and then $u(t)\to x^p(t)$ as $t\to+\infty$. The proof is complete. □

If $F(t,x((t))=r(t)(1-x(t)/K(t))$, then systems (2.1) and (2.2) are in the following form

$$\dot{x}(t)=r(t)x(t)\left[1-\frac{x(t)}{K(t)}\right],\quad t\neq\tau_k,\ \ k\in N, \tag{2.7}$$

$$x(\tau_k^+)=x(\tau_k)(1-E_k),\quad t=\tau_k,\ \ k\in N, \tag{2.8}$$

where $r(t)$ and $K(t)$ are continuous functions and satisfy

$$r(t+T)=r(t),\quad K(t+T)=K(t),$$
$$\tau_{k+q}=\tau_k+T,\quad E_{k+q}=E_k,\ \ 0<E_k<1, \tag{2.9}$$

and E_k is called impulse harvesting effort. The intrinsic rate of change $r(t)$ is related to the periodically changing possibility of regeneration of the species, and the carrying capacity of the system $K(t)$ is related to the periodic change of the resources maintaining the evolution of the population. The jump condition (2.8) reflects the impulse harvesting effect on the population. Let $x(t)$ be a solution of (2.7) and (2.8) with positive initial value $x_0=x(0)>0$, then $x(t)>0$ for all $t\geqslant 0$. Further, it is bounded, that is,

$$x(t)<\max\{x(0),\overline{M}\}, \tag{2.10}$$

where $\overline{M}=\max_{t\in[0,T]}K(t)$.

Further, if we assume

$$\left[\prod_{k=1}^{q}(1-E_k)\right]\mathrm{e}^{\int_0^T r(\tau)\,\mathrm{d}\tau}>1 \tag{2.11}$$

then the conclusions of Theorem 2.1 imply the following results.

Theorem 2.2. *Assume that* (2.9) *and* (2.11) *hold, then there exists a unique positive T-periodic solution* $x^p(t)\in PC_T'$

to systems (2.7) *and* (2.8), *which is expressed as*

$$x^p(t) = \left[e^{\int_0^T r(\tau)\,d\tau} \prod_{k=1}^{q}(1-E_k) - 1 \right] \times \left[\int_t^{t+T} \frac{r(s)}{K(s)} \exp\left(-\int_s^t r(\tau)\,d\tau\right) \times \prod_{t \leqslant \tau_k < s} (1-E_k)\,ds \right]^{-1}. \quad (2.12)$$

In addition, $x^p(t)$ *is globally asymptotically stable for* $x(t)$ *with positive initial value* $x(t_0) = x_0 > 0$ *in the sense that*

$$\lim_{t\to+\infty} |x(t) - x^p(t)| = 0. \quad (2.13)$$

Proof. It is easy to verity that Eq. (2.12) is a T-period solution of Eqs. (2.7) and (2.8). From Theorem 2.1, we can see that (2.13) holds. □

Remark. In Theorem 2.2 if $r(t), K(t) \in PC_T$, the conclusions are also true. Moreover, the solution of systems (2.7) and (2.8) with initial value $x(t_0) = x_0$ can be expressed as

$$x(t) = \left(\frac{1}{x_0} e^{(-\int_{t_0}^t r(\tau)\,d\tau)} \prod_{t_0 \leqslant \tau_k < t} \frac{1}{1-E_k} + \int_{t_0}^t \frac{r(s)}{K(s)} e^{(-\int_s^t r(\tau)\,d\tau)} \prod_{s \leqslant \tau_k < t} \frac{1}{1-E_k}\,ds \right)^{-1}. \quad (2.14)$$

Let η_1, η_2 with $\eta_1 \leqslant \eta_2$ be positive constants. In particular, we can choose $\eta_1 = \min_{t\in[0,T]} x^p(t)$, $\eta_2 = \max_{t\in[0,T]} x^p(t)$. It follows from Theorem 2.2 that there exists a $\bar{t} > 0$ such that for any solution given by (2.14) we have

$$\eta_1 - \varepsilon_0 \leqslant x(t) \leqslant \eta_2 + \varepsilon_0, \quad t > \bar{t},$$

where $\varepsilon_0 > 0$ is sufficiently small. That is, we have the following result.

Corollary 2.1. *Assume* (2.9) *and* (2.11) *hold true*, *then the systems* (2.7) *and* (2.8) *are permanent*, *that is*, *the exploited population modelled by* (2.7) *and* (2.8) *is sustainable development*.

3. Optimal impulsive harvesting policy

In this section, taking a fishery management as an example, we show how to plan harvesting policy in order to sustain fish population at high levels of productivity or economic results, that is, we consider the problem of optimal harvest policy for periodic logistic equation. For this purpose, we first choose the maximum annual biomass yield as the management objective, and assume that $r(t), K(t) \in PC_T$, and are 1-period functions (that is, $T = 1$).

In the rest of paper, we choose the harvesting effort $E_k, k = 1, 2, \ldots, q$ as control variables. Define the admissible set $\mathscr{S} = \{E_k | E_{k+q} = E_k, 0 \leqslant E_k \leqslant 1, k = 1, 2, \ldots, q\}$. We note that $E_k = 0$ implies that there is no harvest at time τ_k, and consequently there exists a unique T-period solution of the systems (2.7) and (2.8) which is globally asymptotically stable (see Fan [10]). $E_k = 1$ means that from time τ_k the population becomes extinction. Without loss of generality, we only investigate the optimal harvesting policy in a system period (e.g., in one year). For the sake of computation, from now on we denote

$$\tau_0 < n < \tau_1 < \tau_2 < \cdots < \tau_q < n+1 < \tau_1 + 1,$$

where n is a positive integer. So the annual impulsive harvesting yield can be expressed as

$$Y_{\{E_k\}_{k=1}^q} = \sum_{k=1}^{q} E_k x(\tau_k). \quad (3.1)$$

What we wish to do is to find $E_k^* \in \mathscr{S} (k = 1, 2, \ldots, q)$ such that $Y_{\{E_k\}_{k=1}^q}$ reaches its maximum, namely to find the optimal impulsive harvesting effort, which is a dynamic optimization problem of a functional. It is only necessary to solve

$$Y_{\{E_k^*\}_{k=1}^q} = \max_{E_k \in \mathscr{S}} Y_{\{E_k\}_{k=1}^q}, \quad (3.2)$$

with constraints

$$\dot{x}(t) = r(t)x(t)\left[1 - \frac{x(t)}{K(t)}\right], \quad t \neq \tau_k,$$

$$x(\tau_k^+) = x(\tau_k)(1-E_k), \quad t = \tau_k, \quad k = 1, 2, \ldots, q.$$

Since the solutions of Eqs. (2.7) and (2.8) with positive initial values, which exist uniquely, are positive and uniformly bounded for all impulsive harvesting effort $E_k, k = 1, 2, \ldots, q$, we can take use of Theorem 5.1 in chapter III of Berkovtiz [7] to see the existence of an optimal control in the class of bounded measurable functions.

Definition 3.1 (*Artstein [3]*). If the control policy $\{E_k, k = 1, 2, \ldots, q\}$ is such that Eqs. (2.7) and (2.8)) have an asymptotically stable periodic solution or equilibrium, then the biomass yield (3.1) is said to be a sustainable yield.

Integrate Eq. (2.7) in any impulsive internal, for example, $t \in (\tau_k, \tau_{k+1}]$, gives

$$x(t) = \left[\frac{1}{x(\tau_k^+)} e^{-\int_{\tau_k}^t r(\tau)\,d\tau} + \int_{\tau_k}^t \frac{r(s)}{K(s)} e^{-\int_s^t r(\tau)\,d\tau}\,ds \right]^{-1}. \quad (3.3)$$

Using Eq. (2.8) yields

$$x(\tau_{k+1}) = \left[\frac{1}{(1-E_k)x(\tau_k)} e^{-\int_{\tau_k}^{\tau_{k+1}} r(\tau)\,d\tau} + \int_{\tau_k}^{\tau_{k+1}} \frac{r(s)}{K(s)} e^{-\int_s^{\tau_{k+1}} r(\tau)\,d\tau}\,ds \right]^{-1}.$$

For convenience, denote $x(k) \triangleq x(\tau_k), k = 1, \ldots, q$. Then the optimal impulsive problem (3.2) can be converted into the following discrete optimization problem subject to certain contains, namely

$$\max_{E_k \in \mathscr{S}} Y_{\{E_k\}_{k=1}^q} = \sum_{k=1}^{q} E_k x(k), \tag{3.4}$$

subject to

$$x(k+1) = \left[\frac{1}{(1-E_k)x(k)} e^{-\int_{\tau_k}^{\tau_{k+1}} r(\tau)\,d\tau} + \int_{\tau_k}^{\tau_{k+1}} \frac{r(s)}{K(s)} e^{-\int_s^{\tau_{k+1}} r(\tau)\,d\tau}\, ds\right]^{-1}. \tag{3.5}$$

In the following, we investigate the optimal harvest problem (3.4)–(3.5) using discrete time optimal control theory. For convenience, denote

$$A = e^{(\int_0^1 r(\tau)\,d\tau)}, \quad D(\tau_k) = e^{(-\int_{\tau_{k-1}}^{\tau_k} r(\tau)\,d\tau)},$$

$$B(\tau_k) = \int_{\tau_{k-1}}^{\tau_k} \frac{r(s)}{K(s)} e^{(-\int_s^{\tau_k} r(\tau)\,d\tau)}\, ds.$$

By applying discrete time optimal control theory, we can obtain the following theorem.

Theorem 3.1. *If*

$$\frac{D^{1/2}(\tau_{k+1})(1-D^{1/2}(\tau_{k+1}))}{1-D^{1/2}(\tau_k)} \frac{B(\tau_k)}{B(\tau_{k+1})} \leqslant 1,$$

then there exists a unique optimal impulsive harvest effort $\{E_k^*\}_{k=1}^q$, *namely*,

$$E_k^* = 1 - \frac{D^{1/2}(\tau_{k+1})(1-D^{1/2}(\tau_{k+1}))}{1-D^{1/2}(\tau_k)} \frac{B(\tau_k)}{B(\tau_{k+1})}, \tag{3.6}$$

which solves uniquely the dynamic optimization problem (3.2).

Moreover, *the corresponding optimal population level is*

$$x^{*p}(t) = \left[e^{-\int_{\tau_k}^{t} r(\tau)\,d\tau} B(\tau_{k+1})[D^{1/2}(\tau_{k+1}) - D(\tau_{k+1})]^{-1} + \int_{\tau_k}^{t} \frac{r(s)}{K(s)} e^{(-\int_s^t r(\tau)\,d\tau)}\, ds\right]^{-1}, \tag{3.7}$$

where $t \in (\tau_k, \tau_{k+1}]$, $k = 1, 2, \ldots, q$.

The maximum annual harvesting yield reads

$$Y_{\{E_k^*\}_{k=1}^q} = \sum_{k=1}^{q} \frac{(1-D^{1/2}(\tau_k))^2}{B(\tau_k)}, \tag{3.8}$$

and further, *which is the maximum annual-sustainable yield.*

Proof. In order to utilize optimal control theory (see [11,14]) directly, we shall minimize the function

$$\bar{Y}_{\{E_k\}_{k=1}^q} = -\sum_{k=1}^{q} E_k x(k).$$

Then solving (3.2) is equivalent to solve the equation

$$\bar{Y}_{\{E_k^*\}_{k=1}^q} = \min_{E_k \in \mathscr{S}} \bar{Y}_{\{E_k^*\}_{k=1}^q}.$$

The Hamilton function is

$$H(x(k), E_k, \lambda(k+1), \tau_k) = -E_k x(k) + \lambda(k+1) \left[\frac{D(\tau_{k+1})}{(1-E_k)x(k)} + B(\tau_{k+1})\right]^{-1},$$

where $\lambda(k+1)$ is the costate variable.

If $\{E_k^*, k = 1, 2, \ldots, q\}$ is an optimal control sequence and $\{x^*(k), k = 1, 2, \ldots, q\}$ is the corresponding optimal trajectory (the optimal population level). Then it follows from [11, Theorem 2.6.1] that the necessary conditions for optimality are

$$\lambda(k) = \frac{\partial H}{\partial x^*(k)},$$

$$\frac{\partial H}{\partial E_k^*} = 0,$$

which gives

$$\lambda(k) = -E_k^* + \lambda(k+1)\left[\frac{D(\tau_{k+1})}{(1-E_k^*)x^*(k)} + B(\tau_{k+1})\right]^{-2} \times D(\tau_{k+1})[(1-E_k^*)(x^*(k))^2]^{-1},$$

$$x^*(k) = -\lambda(k+1)\left[\frac{D(\tau_{k+1})}{(1-E_k^*)x^*(k)} + B(\tau_{k+1})\right]^{-2} \times D(\tau_{k+1})[x^*(k)(1-E_k^*)^2]^{-1}. \tag{3.9}$$

It follows form the second equation of (3.9) that we have

$$\lambda(k+1) = -\left[\frac{D(\tau_{k+1})}{(1-E_k^*)x^*(k)} + B(\tau_{k+1})\right]^2 \times (D(\tau_{k+1}))^{-1}[x^*(k)(1-E_k^*)]^2.$$

Substituting above equality into the first equation of (3.9) yields $\lambda(k) = -1$. Again, combining with (3.5) gives

$$\frac{(1-E_k^*)x^*(k)}{x^*(k+1)} = e^{(-(1/2)\int_{\tau_k}^{\tau_{k+1}} r(\tau)\,d\tau)} = D^{1/2}(\tau_{k+1}), \quad k = 1, 2, \ldots, q. \tag{3.10}$$

Difference Eq. (3.5) gives

$$\frac{1}{x^*(k+1)} = \frac{D(\tau_{k+1})}{(1-E_k^*)x^*(k)} + B(\tau_{k+1}). \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$x^*(k) = \frac{1-D^{1/2}(\tau_k)}{B(\tau_k)}, \quad k = 1, 2, \ldots, q. \tag{3.12}$$

Substituting $x^*(k)$ into (3.10) gives

$$E_k^* = 1 - D^{1/2}(\tau_{k+1})\frac{x^*(k+1)}{x^*(k)} = 1 - \frac{D^{1/2}(\tau_{k+1})(1-D^{1/2}(\tau_{k+1}))}{D^{1/2}(\tau_k)} \frac{B(\tau_k)}{B(\tau_{k+1})},$$

$$k = 1, 2, \ldots, q \tag{3.13}$$

which implies that E_k^* is unique and less than one. The assumption of the theorem implies that $E_k^* \geqslant 0$, then $E_k^* \in \mathscr{S}$.

Therefore, by the uniqueness of solution of Eq. (3.9), E_k^* is the unique solution of Eq. (3.4), which is the optimal impulsive harvesting effort.

Further, (3.10) and (3.12) give

$$(1-E_k^*)x^*(k) = D^{1/2}(\tau_{k+1})x^*(k+1) = \frac{D^{1/2}(\tau_{k+1})(1-D^{1/2}(\tau_{k+1}))}{B(\tau_{k+1})}. \tag{3.14}$$

For any $t>0$, say, $t \in (\tau_k, \tau_{k+1}]$, then from (3.3) we get

$$x^*(t) = \left[\frac{1}{(1-E_k^*)x^*(k)} e^{(-\int_{\tau_k}^t r(\tau)\,d\tau)} + \int_{\tau_k}^t \frac{r(s)}{K(s)} e^{(-\int_s^t r(\tau)\,d\tau)}\,ds\right]^{-1}. \tag{3.15}$$

Substituting (3.14) into (3.15) yields the optimal population level

$$x^*(t) = \left[e^{(-\int_{\tau_k}^t r(\tau)\,d\tau)} B(\tau_{k+1})[D^{1/2}(\tau_{k+1}) - D(\tau_{k+1})]^{-1} + \int_{\tau_k}^t \frac{r(s)}{K(s)} e^{(-\int_s^t r(\tau)\,d\tau)}\,ds\right]^{-1},$$

$$t \in (\tau_k, \tau_{k+1}], \quad k=1,2,\ldots,q. \tag{3.16}$$

It is easy to show $B(\tau_k+1)=B(\tau_k)$, $D(\tau_k+1)=D(\tau_k)$ for any τ_k from the definitions of $B(\tau_k)$ and $D(\tau_k)$. Observing the periodicity of τ_k given by (2.9) yields

$$D(\tau_{q+1}) = D(\tau_1+1) = D(\tau_1),$$

$$B(\tau_{q+1}) = B(\tau_1+1) = B(\tau_1),$$

then the maximum annual biomass yield is

$$\begin{aligned} Y_{\{E_k^*\}_{k=1}^q} &= \sum_{k=1}^q E_k^* x^*(k) \\ &= \sum_{k=1}^q \left[\frac{1-D^{1/2}(\tau_k)}{B(\tau_k)} - \frac{D^{1/2}(\tau_{k+1})(1-D^{1/2}(\tau_{k+1}))}{B(\tau_{k+1})}\right] \\ &= \sum_{k=1}^q \frac{(1-D^{1/2}(\tau_k))^2}{B(\tau_k)}. \end{aligned} \tag{3.17}$$

In addition, it is easy to prove the optimal population level $x^*(t)$ corresponding to the optimal impulsive harvesting effort E_k^* is the unique positive 1-period solution of systems (2.7) and (2.8). In fact, for any $t \in (\tau_k, \tau_{k+1}]$, then $t+1 \in (\tau_k+1, \tau_{k+1}+1]$, it follows from (3.16) that

$$\begin{aligned} x^*(t+1) &= \Bigg[e^{(-\int_{\tau_k+1}^{t+1} r(\tau)\,d\tau)} B(\tau_{k+1}+1)[D^{1/2}(\tau_{k+1}+1) - D(\tau_{k+1}+1)]^{-1} + \int_{\tau_k+1}^{t+1} \frac{r(s)}{K(s)} \times e^{(-\int_s^{t+1} r(\tau)\,d\tau)}\,ds\Bigg]^{-1} \\ &= \left[e^{(-\int_{\tau_k}^t r(\tau)\,d\tau)} B(\tau_{k+1})[D^{1/2}(\tau_{k+1}) - D(\tau_{k+1})]^{-1} + \int_{\tau_k}^t \frac{r(s)}{K(s)} e^{(-\int_s^t r(\tau)\,d\tau)}\,ds\right]^{-1} = x^*(t). \end{aligned}$$

Further,

$$\begin{aligned} & e^{(\int_0^1 r(\tau)\,d\tau)} \prod_{k=1}^q (1-E_k^*) \\ &= A \prod_{k=1}^q \frac{D^{1/2}(\tau_{k+1})(1-D^{1/2}(\tau_{k+1}))}{1-D^{1/2}(\tau_k)} \frac{B(\tau_k)}{B(\tau_{k+1})} \\ &= A e^{(-(1/2)\int_{\tau_1}^{\tau_q+1} r(\tau)\,d\tau)} = A e^{(-(1/2)\int_{\tau_1}^{\tau_1+1} r(\tau)\,d\tau)} \\ &= A^{1/2} > 1, \end{aligned}$$

which implies (2.11) holds true. It follows from Theorem 2.2 that there exists a unique positive 1-period solution of Eqs. (2.7) and (2.8) which is globally asymptotically stable. The optimal population level $x^*(t)$ corresponding to the optimal harvest effort E_k^* $(k=1,2,\ldots,q)$ is the 1-period solution of Eqs. (2.7) and (2.8), hence it is globally asymptotically stable. Thus from Definition 3.1, the maximum annual biomass yield $Y_{\{E_k^*\}_{k=1}^q}$ given by (3.17) is the maximum annual-sustainable yield. This completes the proof. □

The results show that the optimal harvesting policy obtained here can not only maximize the annual biomass yield but also make the fish population permanent (since the optimal population level $x^*(k)$ corresponding to optimal impulsive harvesting effort $\{E_k^*\}_{k=1}^q$ is truly the periodic solution of (2.7) and (2.8) which is globally asymptotically stable), then the maximum biomass yield is in fact the MSY. Hence, this type of harvesting policy has a built-in stability mechanism, and is of higher priority to help reduce overfishing, which is of biological meanings.

Remark 3.1. For an optimal impulsive harvesting problem, it must be noted that there are two kinds of variable vectors, impulsive functions ($x(\tau_k)(1-E_k)$ given in Eq. (2.8)) and impulsive moments ($\tau_k, k=1,2,\ldots,q$ in Eqs. (2.7) and (2.8)), could be controlled. When choosing impulsive functions as control variables (at the same time, assuming $\tau_k, k=1,2,\ldots,q$ are fixed), and the maximum annual-sustainable yield as management objective, we have investigated the optimal harvesting policy for periodic logistic equation. While choosing impulsive moments as control variable (assuming $E_k, k=1,2,\ldots,q$ are fixed), and the same management objective, it is not difficult to see that we can also work out the optimal impulsive moments, the optimal population level and the corresponding annual-sustainable yield by using the ideas and methods similar to the above arguments.

Remark 3.2. For the computation of an optimal control problem (3.2) or (3.4)–(3.5), we note that the optimal control software package, MISER 3.2 can be used (see Jennings et al. [13] for more details). In addition, Liu et al. [18] provided a computational method for solving a general continuous time impulsive optimal control problem.

4. Comparing the results with those in continuous optimal harvesting policy

It is interesting to compare the periodic logistic equation with impulsive harvest analyzed here with the analogous ordinary differential equation with continuous harvest analyzed in [10], which is modelled by system (1.3). From the point view of constructing model, the models here are more natural and realistic due to taking the discontinuity of human exploit activities into account. It was shown in [10] that, in order to gain the maximum annual-sustainable yield, the optimal harvesting effort should be

$$E_* = \frac{r(t)}{2} - \frac{\dot K(t)}{K(t)} \quad \left(\text{if } r(t) \geqslant \frac{2\dot K(t)}{K(t)}\right).$$

The maximum annual-sustainable yield, corresponding to E^* is

$$Y_* = \frac{1}{4}\int_0^1 r(t)K(t)\,\mathrm{d}t,$$

and the corresponding optimal population level is given by

$$x_*(t) = \tfrac{1}{2}K(t).$$

In the following, we try to compare the optimal results of impulsive system with the results of continuous system. It must be noted that the impulsive differential equation can be reduced the corresponding ordinary differential equation if the length of the maximum impulsive intervals

$$\Delta\tau = \max\{\Delta\tau_k : \Delta\tau_k = \tau_{k+1} - \tau_k, k = 0,1,2,\ldots,q\}$$

tends to zero (which implies impulsive times in a period $q \to +\infty$). For this purpose, we assume that $r(t)$, $K(t)$ are continuous functions and then investigate how the maximum annual-sustainable yield $Y_{\{E_k^*\}_{k=1}^q}$ and the corresponding optimal population level $x^*(t)$ change as $\Delta\tau \to 0$.

Formula (3.12) can be written as

$$x^*(\tau_k + \Delta\tau_k) = \frac{1 - D^{1/2}(\tau_k+\Delta\tau_k)}{B(\tau_k+\Delta\tau_k)} = \frac{1 - \mathrm{e}^{(-(1/2)\int_{\tau_k}^{\tau_k+\Delta\tau_k} r(\tau)\,\mathrm{d}\tau)}}{\int_{\tau_k}^{\tau_k+\Delta\tau_k} \frac{r(s)}{K(s)} \mathrm{e}^{(-\int_s^{\tau_k+\Delta\tau_k} r(\tau)\,\mathrm{d}\tau)}\,\mathrm{d}s}.$$

Then

$$\begin{aligned}
&\lim_{\Delta\tau_k\to 0} x^*(\tau_k+\Delta\tau_k)\\
&= \lim_{\Delta\tau_k\to 0} \frac{1 - \mathrm{e}^{(-(1/2)\int_{\tau_k}^{\tau_k+\Delta\tau_k} r(\tau)\,\mathrm{d}\tau)}}{\int_{\tau_k}^{\tau_k+\Delta\tau_k} \frac{r(s)}{K(s)} \mathrm{e}^{(-\int_s^{\tau_k+\Delta\tau_k} r(\tau)\,\mathrm{d}\tau)}\,\mathrm{d}s}\\
&= \lim_{\Delta\tau_k\to 0} \\
&\frac{\mathrm{e}^{(-(1/2)\int_{\tau_k}^{\tau_k+\Delta\tau_k} r(\tau)\,\mathrm{d}\tau)}\frac{1}{2}r(\tau_k+\Delta\tau_k)}{-\int_{\tau_k}^{\tau_k+\Delta\tau_k} \frac{r(s)}{K(s)} \mathrm{e}^{(-\int_s^{\tau_k+\Delta\tau_k} r(\tau)\,\mathrm{d}\tau)} r(\tau_k+\Delta\tau_k)\,\mathrm{d}s + \frac{r(\tau_k+\Delta\tau_k)}{K(\tau_k+\Delta\tau_k)}}\\
&= \frac{K(\tau_k)}{2}.
\end{aligned} \tag{4.1}$$

Moreover, as $\Delta\tau_k$ tends to zero and q tends to infinity in one period, (3.17) becomes

$$\begin{aligned}
&\sum_{k=1}^{\infty} \lim_{\Delta\tau_k\to 0} \frac{(1 - D^{1/2}(\tau_k+\Delta\tau_k))^2}{B(\tau_k+\Delta\tau_k)}\\
&= \sum_{k=1}^{\infty} \lim_{\Delta\tau_k\to 0} \frac{1 - D^{1/2}(\tau_k+\Delta\tau_k)}{B(\tau_k+\Delta\tau_k)} \frac{1 - D^{1/2}(\tau_k+\Delta\tau_k)}{\Delta\tau_k}\Delta\tau_k\\
&= \sum_{k=1}^{\infty} \frac{1}{4} K(\tau_k) r(\tau_k)\Delta\tau_k\\
&= \frac{1}{4}\int_0^1 K(t)r(t)\,\mathrm{d}t,
\end{aligned} \tag{4.2}$$

which implies that the maximum annual-sustainable yield and the correspond optimal population level tend to their counterparts of periodic logistical model with continuous harvest as the length of the maximum impulsive intervals tend to zero in one period. This shows that the results obtained in this paper extend and generalize the classical results of Fan and Clark for renewable resources management.

In fact, in practice we can only take up a finite times impulsive harvest in a period (e.g., in one year) so as to obtain the maximum annual-sustainable yield which closely equals that of continuous harvesting policy, and also obtain the corresponding optimal population level which approximately equals that of continuous harvesting policy (see Fig. 1) in the sense of mean. As a numerical example, we let

$$r(t) = 2 + \tfrac{1}{5}\cos(2\pi t), \qquad K(t) = 5 + \tfrac{1}{5}\sin(2\pi t),$$

which are continuous functions with 1-period. According to the results of Fan et al. [10], the maximum annual-sustainable yield is $\frac{1}{4}\int_0^1 r(t)K(t)\,\mathrm{d}t = 2.5$, the corresponding optimal population level is $\frac{1}{2}K(t)$. If we assume that there are 40 impulsive moments in (4, 5] with the equivalent length of all impulsive intervals, then the optimal impulsive harvesting efforts can be obtained step by step in Maple, the corresponding maximum

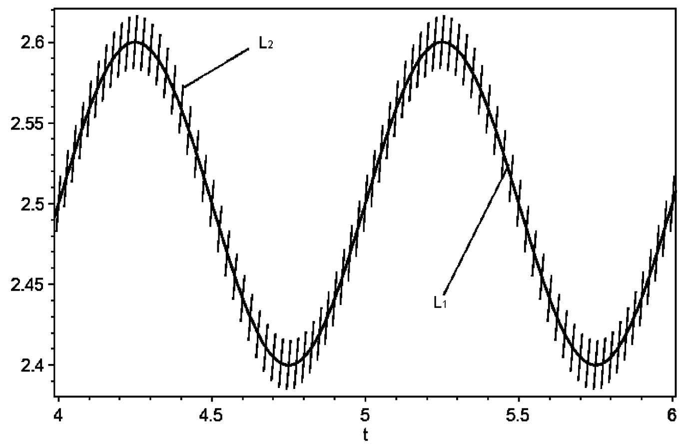

Fig. 1. The optimal population level $L_1 = K(t)/2$ under the continuous harvesting policy and the optimal population level $L_2 = x^*(t)$ under the impulsive harvesting policy.

annual-sustainable yields $Y_{\{E_k^*\}_{k=1}^{40}} \approx 2.49987$, which supports the above conclusion. Hence, impulsive harvesting policy can reduce the cost of harvest. Fig. 1 describes the case of eighties impulsive moments in (4, 6]. In Fig. 1, the continuous curve L_1 denotes the optimal population level under continuous harvest policy, that is, $L_1 = K(t)/2$. The optimal population level $x^*(t)$ given by (3.16) corresponding to optimal impulsive harvesting policy yields the following piece-wise continuous function

$$x^*(t)=$$
$$\begin{cases} x^*(t;\tau_0, x^*(\tau_0^+)) \triangleq L_2^1, & t \in (\tau_0, \tau_1] = \left(4 - \frac{1}{80}, 4 + \frac{1}{80}\right], \\ x^*(t;\tau_1, x^*(\tau_1^+)) \triangleq L_2^2, & t \in (\tau_1, \tau_2] = \left(4 + \frac{1}{80}, 4 + \frac{3}{80}\right], \\ \vdots & \vdots \\ x^*(t;\tau_{40}, x^*(\tau_{40}^+)) \triangleq L_2^{41}, & t \in (\tau_{40}, \tau_1 + 1] = \left(4 + \frac{79}{80}, 5 + \frac{1}{80}\right]. \end{cases}$$

It follows from Fig. 1 that each graph L_2^k in subinterval $(\tau_{k-1}, \tau_k]$ $(k = 1, 2, \ldots, 41)$ are symmetrically distributed near the curve L_1, which confers to reality.

Acknowledgements

The authors thank the referees for their careful reading of the original manuscript and many valuable comments and suggestions.

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