satisfies Lipschitz condition. Based on the result on the existence and uniqueness of solution to CSSP, necessary and sufficient conditions for generalized quadratic stability is obtained by using S-procedure approach and matrix inequality technique. The proposed convex optimization approach guarantees global exponential stability and simultaneously maximizes the tolerable perturbation bound. The approach presented in this note has improved and generalized the results and techniques in the literature. The assumptions of index one and single equilibrium systems will be relaxed and investigated in our future work.

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Optimal Estimation for Continuous-Time Systems With Delayed Measurements

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***Abstract*—This note focuses on the traditional problem of the Kalman filtering for linear continuous-time systems. Although the problem has been studied widely in the past decades, little work has been done for the time-delayed systems and some fundamental problems remain to be solved. This note proposes a new tool, namely, *reorganization innovation analysis approach*, to investigate the filtering problem for systems with delayed measurements. The Kalman filter is given in terms of the solution of standard Riccati equations. The performance is clearly demonstrated through analytical results and simulation. The solved problem in this note is related with some more complicated problems such as H_∞ fixed-lag smoothing, H_∞ control with preview and control with input delays.**

***Index Terms*—Continuous-time system, Kalman filter, reorganized innovation analysis, Riccati equations, time-delayed systems.**

I. INTRODUCTION

Linear estimation has important applications in many fields, such as communication, control, econometrics and signal processing, etc. The problem has attracted significant attention in the past 50 years [4], [6], [7]. There are two main approaches to the linear estimation. One is the minimum variance estimation which is termed as H_2 [5], [10], [16]. The other is the H_∞ estimation which has emerged as an alternative since the 1980s [6], [11]. For the systems without delay, most of the estimation or control problems under the two performances have been well studied. In the case of time-delay, however, the estimation or control problem is much more complicated and some problems remain to be investigated.

The Kalman filter (H_2 estimation), which addresses the minimization of filtering error covariance, has been a classical tool in signal processing, communication and control applications. It has been widely studied via Riccati equation approach [7]. However, the Kalman filtering formulation is only applicable to the standard systems without delays. In the time delays context, the optimal estimation has been studied via partial differential equation (PDE) [10] for continuous-time systems or state augmentation method for discrete-time systems. Note that the PDE is very difficult to be solved in general (in fat it is impossible to have a analytical solution) and the state augmentation leads to very expensive cost. Very recently, [1] and [2] studied the estimation and control problem for observation-delay systems via differential Riccati-type equations.

In this note, we consider the minimum mean square error (MMSE) estimation problem for linear continuous time varying systems with current and time-delay measurements. Such problem has obvious applications to many engineering problems. Furthermore, the problem has been shown to be related with some complicated problems such as H_∞ fixed-lag smoothing [14], [15], preview control [12] and H_∞ control with control input signal delays [8], [9]. Our aim is to present the

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Kalman filtering formulation for such problem in terms of the differential Riccati equation. It will be shown that, different from the standard delay-free system, the Kalman filtering formulation for the time delay systems consists of two standard Kalman filters with the same dimension as the original systems.

The rest of the note is organized as follows. The problem formulation is given in Section II. The main results for the H_2 filter is presented in Section III. In Section IV, a numerical example is constructed to illustrate the main result. Section V contains some concluding remarks.

II. Problem Formulation

We consider the following linear system for H_2 estimation problem:

$$\dot{\mathbf{x}}(t) = \Phi(t)\mathbf{x}(t) + \Gamma(t)\mathbf{u}(t) \quad (1)$$

where $\mathbf{x}(t) \in R^n$ and $\mathbf{u}(t) \in R^r$, represent the state and input noise, respectively, $\Phi(t)$ and $\Gamma(t)$ are bounded time-varying matrices with appropriate dimensions. Assume that the state $\mathbf{x}$ is observed by different systems with delays which are described by

$$\mathbf{y}(t) = H(t)\mathbf{x}(t) + \mathbf{v}(t) \quad (2)$$

$$\mathbf{y}_1(t) = H_1(t)\mathbf{x}(t-d) + \mathbf{v}_1(t), \qquad d > 0 \quad (3)$$

where $\mathbf{y}(t) \in R^p$ and $\mathbf{y}_1(t) \in R^{p_1}$ are respectively the current and the delayed output measurement, $\mathbf{v}(t) \in R^p$ and $\mathbf{v}_1(t) \in R^{p_1}$ are the related the measurement noises. It is assumed that the input $\mathbf{u}$ and measurement noise $\mathbf{v}(t)$ are from $L_2[0,T]$, and the delayed measurement noise $\mathbf{v}_1(t)$ is from $L_2[d,T]$, while $T > 0$ is the time-horizon of filtering. Let $\mathbf{y}_d(t)$ denote the observation of the system (2), (3) at time t and $\mathbf{v}_d(t)$ the related observation noise at time t, then we have

$$\mathbf{y}_d(t) = \begin{cases} \mathbf{y}(t), & 0 \le t < d \\ col\left\{\mathbf{y}(t),\ \mathbf{y}_1(t)\right\}, & t \ge d \end{cases} \quad (4)$$

$$\mathbf{v}_d(t) = \begin{cases} \mathbf{v}(t), & 0 \le t < d \\ col\left\{\mathbf{v}(t),\ \mathbf{v}_1(t)\right\}, & t \ge d. \end{cases} \quad (5)$$

Now, we make the following standard assumptions for the systems (1)–(3)

Assumption 2.1: The initial state $\mathbf{x}(0)$ and the noises $\mathbf{u}(t)$, $\mathbf{v}(t)$, $\mathbf{v}_1(t)$ are mutually uncorrelated white noises with zero means and known covariance matrices as

$$\langle \mathbf{x}(0),\ \mathbf{x}(0)\rangle = \Pi_0 \quad (6)$$

$$\langle \mathbf{u}(t), \mathbf{u}(\tau)\rangle = Q_u(t)\delta(t-\tau) \quad (7)$$

$$\langle \mathbf{v}(t), \mathbf{v}(\tau)\rangle = Q_v(t)\delta(t-\tau) \quad (8)$$

$$\langle \mathbf{v}_1(t), \mathbf{v}_1(\tau)\rangle = Q_{v_1}(t)\delta(t-\tau). \quad (9)$$

Problem P: ***Given the observation*** $\{\{\mathbf{y}_d(\tau)\}|_{0\le\tau\le t}\}$***, find a linear least mean square error estimator*** $\hat{\mathbf{x}}(t|t)$ ***of*** $\mathbf{x}(t)$.

In the following, for the convenience of discussions we will denote that

$$t_1 \triangleq t - d.$$

III. Main Results

The basic idea to deal with the time delay in this note is to reorganize the observations from different channel as new delay-free observations, and introduce the innovation associated with the observations. The optimal filter is then derived by using the reorganization innovation and the projection formula. For the simplicity of discussions we first suppose that the time $t > d$. The case of $0 \le t \le d$ will be considered later.

A. Reorganization Innovation

We first define the reorganization observation in the following lemma.

Lemma 3.1: The linear space of $\mathcal{L}\{\mathbf{y}_d(\tau), 0 \le \tau \le t\}$ is equivalent to

$$\mathcal{L}\left\{\mathcal{Y}_2(\tau)|_{0\le\tau\le t_1}\ \ \mathcal{Y}_1(\tau)|_{t_1<\tau\le t}\right\} \quad (10)$$

where $\mathcal{Y}_2(\tau)$ and $\mathcal{Y}_1(\tau)$ are the reorganized new observations as

$$\mathcal{Y}_2(\tau) \triangleq \begin{bmatrix} \mathbf{y}(\tau) \\ \mathbf{y}_1(\tau+d) \end{bmatrix} \quad (11)$$

$$\mathcal{Y}_1(\tau) \triangleq \mathbf{y}(\tau) \quad (12)$$

satisfy that

$$\mathcal{Y}_i(\tau) = \mathcal{H}_i(\tau)\mathbf{x}(\tau) + \mathcal{V}_i(\tau), \qquad i = 1, 2 \quad (13)$$

with

$$\mathcal{H}_2(\tau) \triangleq \begin{bmatrix} H(\tau) \\ H_1(\tau+d) \end{bmatrix} \quad \mathcal{H}_1(\tau) \triangleq H(\tau). \quad (14)$$

$$\mathcal{V}_2(\tau) \triangleq \begin{bmatrix} \mathbf{v}(\tau) \\ \mathbf{v}_1(\tau+d) \end{bmatrix} \quad \mathcal{V}_1(\tau) \triangleq \mathbf{v}(\tau). \quad (15)$$

Moreover, $\mathcal{V}_i(\tau)$ is white noise with zero mean and covariance matrix as

$$Q_{\mathcal{V}_2}(\tau) = \begin{bmatrix} Q_v(\tau) & 0 \\ 0 & Q_{v_1}(\tau+d) \end{bmatrix} \quad Q_{\mathcal{V}_1}(\tau) = Q_v(\tau). \quad (16)$$

Proof: The proof is straightforward and omitted.

In the above, $\mathcal{Y}_2(\tau)$ and $\mathcal{Y}_1(\tau)$ are the reorganization observations. Now we introduce the innovation associated with the reorganization observation.

Definition 3.1: Consider the linear space of (10), for any $s > t_1 = t - d$ denote

$$\mathbf{w}_1(s) \triangleq \mathcal{Y}_1(s) - \hat{\mathcal{Y}}_1(s) \quad (17)$$

where $\hat{\mathcal{Y}}_1(s)$ is the projection of $\mathcal{Y}_1(s)$ onto linear space

$$\mathcal{L}\left\{\mathcal{Y}_2(\tau)|_{0\le\tau\le t_1}\quad \mathcal{Y}_1(\tau)|_{t_1<\tau<s}\right\}. \quad (18)$$

For $0 \le s \le t_1$, denote

$$\mathbf{w}_2(s) \triangleq \mathcal{Y}_2(s) - \hat{\mathcal{Y}}_2(s) \quad (19)$$

where $\hat{\mathcal{Y}}_2(s)$ is the projection of $\mathcal{Y}_2(s)$ onto linear space

$$\mathcal{L}\left\{\mathcal{Y}_2(\tau)|_{0\le\tau<s}\right\}. \quad (20)$$

In the above, $\mathbf{w}_1(s)$ and $\mathbf{w}_2(s)$ are the prediction error of the reorganization observation. It is easy to observe that $\mathbf{w}_1(s)$ and $\mathbf{w}_2(s)$ have the following relationships as:

$$\mathbf{w}_2(s) = \mathcal{H}_2(s)\tilde{\mathbf{x}}(s,2) + \mathcal{V}_2(s), \qquad 0 \le s \le t_1 \quad (21)$$

$$\mathbf{w}_1(s) = \mathcal{H}_1(s)\tilde{\mathbf{x}}(s,1) + \mathcal{V}_1(s), \qquad s > t_1 \quad (22)$$

where

$$\tilde{\mathbf{x}}(s,2) = \mathbf{x}(s) - \hat{\mathbf{x}}(s,2), \qquad 0 \le s \le t_1 \quad (23)$$

$$\tilde{\mathbf{x}}(s,1) = \mathbf{x}(s) - \hat{\mathbf{x}}(s,1), \qquad s > t_1 \quad (24)$$

while $\hat{\mathbf{x}}(s,2)$ is the projection of $\mathbf{x}(s)$ onto linear space of (20) and $\hat{\mathbf{x}}(s,1)$ is the projection of $\mathbf{x}(s)$ onto linear space of (18).

Lemma 3.2: The stochastic process $\mathbf{w}$ defined in Definition 3.1 is mutually uncorrelated and

$$\left\{\mathbf{w}_2(\tau)|_{0\le\tau\le t_1}\quad \mathbf{w}_1(\tau)|_{t_1<\tau\le t}\right\} \quad (25)$$

spans the same linear space as $\mathcal{L}\{\mathbf{y}_d(\tau),\ 0 \le \tau \le t\}$.

Proof: It is readily seen from (19) that $\mathbf{w}_2(s)$ for $s \le t_1$ (or $\mathbf{w}_1(s)$, $s > t_1$) is a linear combination of the observations

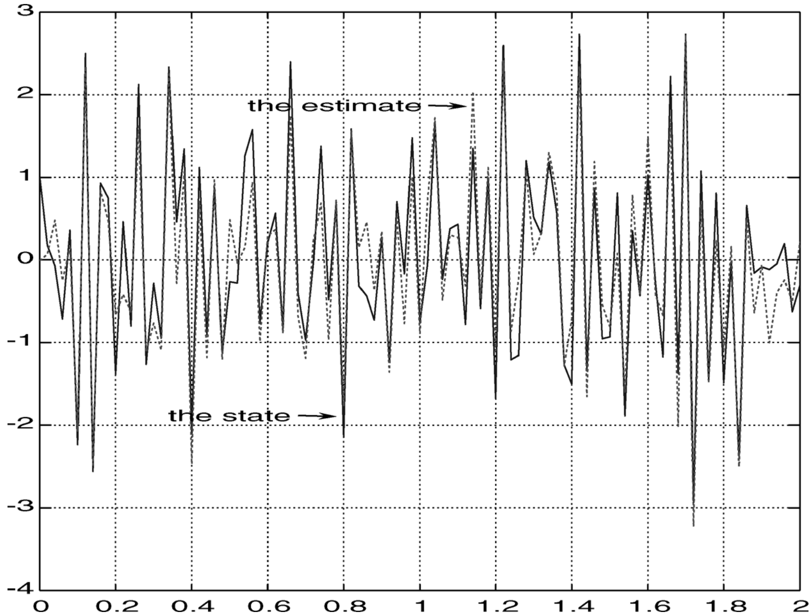

Fig. 1. True state $\hat{\mathbf{x}}_1(t|t)$ and its estimation value with delay.

$\{\mathcal{Y}_2(\tau)|_{0\le\tau\le s}\}$ (or $\{\mathcal{Y}_2(\tau)|_{0\le\tau\le t_1};\ \mathcal{Y}_1(\tau)|_{t_1<\tau\le s}\}$). Conversely, $\mathcal{Y}_2(s)$, $s \le t_1$, (or $\mathcal{Y}_1(s)$, $s > t_1$) can be given in terms of a linear combination of $\mathbf{w}_2(\tau)|_{0\le\tau<s}$ (or $\{\mathbf{w}_2(\tau)|_{0\le\tau\le t_1};\ \mathbf{w}_1(\tau)|_{t_1<\tau<s}\}$). Thus, $\{\mathbf{w}_2(\tau)|_{0\le\tau\le t_1};\ \mathbf{w}_1(\tau)|_{t_1<\tau\le t}\}$ spans the same linear space as $\mathcal{L}\{\mathcal{Y}_2(\tau)|_{0\le\tau\le t_1};\ \mathcal{Y}_1(\tau)|_{t_1<\tau\le t}\}$ or equivalently $\mathcal{L}\{\mathbf{y}_d(\tau)|_{0\le\tau\le t}\}$.

Next, we show that $\{\mathbf{w}_i(\cdot), i = 1,2\}$ is an mutually uncorrelated sequence. In fact, for any $s > t_1$ and $\tau \le t_1$ where $t = t - d_1$, it follows from (17)–(19) that

$$\mathcal{E}\left[\mathbf{w}_1(s)\mathbf{w}_2(\tau)'\right] = \mathcal{H}_1(s)\mathcal{E}\left[\tilde{\mathbf{x}}(s,1)\mathbf{w}_2(\tau)'\right] + \mathcal{E}\left[\boldsymbol{\mathcal{V}}_1(s)\mathbf{w}_2(\tau)'\right]. \tag{26}$$

Note that $\mathcal{E}[\mathcal{V}_1(s)\mathbf{w}_2(\tau)'] = 0$. Since $\tilde{\mathbf{x}}(s,1)$ is the state prediction error, it follows that $\mathcal{E}[\tilde{\mathbf{x}}(s,1)\mathbf{w}_2(\tau)'] = 0$, and thus $\mathcal{E}[\mathbf{w}_1(s)\mathbf{w}_2(\tau)'] = 0$, which implies that $\mathbf{w}_2(\tau)$, $(0 \le \tau \le t_1)$ is uncorrelated with $\mathbf{w}_1(s)$, $(s > t_1)$. Similarly, it can be verified that $\mathbf{w}_2(s)$ is uncorrelated with $\mathbf{w}_2(\tau)$ for $s \ne \tau$ and $\mathbf{w}_1(s_0)$ is uncorrelated with $\mathbf{w}_1(\tau_0)$ for $s_0 \ne \tau_0$. Hence, $\{\mathbf{w}_2(\tau)|_{0\le\tau\le t_1};\ \mathbf{w}_1(\tau)|_{t_1<\tau\le t}\}$ is an innovation sequence. This completes the proof of the lemma.

$\mathbf{w}_i(\tau)$, $i = 1,2$, is termed as *reorganized innovation*, which plays important role for deriving the optimal estimator.

B. Riccati Equation

Given time instant t and $t_1 = t - d$, denote

$$\mathcal{P}_2(\tau) \triangleq \mathcal{E}\left[\tilde{\mathbf{x}}(\tau,2)\tilde{\mathbf{x}}(\tau,2)'\right], \qquad 0 \le \tau \le t_1 \tag{27}$$

$$\mathcal{P}_1(\tau) \triangleq \mathcal{E}\left[\tilde{\mathbf{x}}(\tau,1)\tilde{\mathbf{x}}(\tau,1)'\right], \qquad \tau > t_1 \tag{28}$$

where $\mathcal{P}_2(\tau)$ and $\mathcal{P}_1(\tau)$ are the covariance matrices of prediction error of system state. We shall show that $\mathcal{P}_2(\tau)$ and $\mathcal{P}_1(\tau)$ satisfy Riccati equation.

Theorem 3.1:

1) $\mathcal{P}_2(\tau)$ is the solution of the following Riccati equation:

$$\frac{d\mathcal{P}_2(\tau)}{d\tau} = \Phi(\tau)\mathcal{P}_2(\tau) + \mathcal{P}_2(\tau)\Phi'(\tau) - \mathcal{K}_2(\tau)Q_{\mathcal{V}_2}(\tau)\left[\mathcal{K}_2(\tau)\right]' + \Gamma(\tau)Q_u(\tau)\Gamma'(\tau) \quad \mathcal{P}_2(0) = \Pi_0 \tag{29}$$

where

$$\mathcal{K}_2(\tau) = \mathcal{P}_2(\tau)\mathcal{H}_2'(\tau)Q_{\mathcal{V}_2}^{-1}(\tau). \tag{30}$$

2) $\mathcal{P}_1(\tau)$ for $\tau > t_1$ is the solution of the following Riccati equation:

$$\frac{d\mathcal{P}_1(\tau)}{d\tau} = \Phi(\tau)\mathcal{P}_1(\tau) + \mathcal{P}_1(\tau)\Phi'(\tau) - \mathcal{K}_1(\tau)Q_{\mathcal{V}_1}(\tau)\left[\mathcal{K}_1(\tau)\right]' \times\Gamma(\tau)Q_u(\tau)\Gamma'(\tau) \quad \mathcal{P}_1(t_1) = \mathcal{P}_2(t_1) \tag{31}$$

where

$$\mathcal{K}_1(\tau) = \mathcal{P}_1(\tau)\mathcal{H}_1'(\tau)Q_{\mathcal{V}_1}^{-1}(\tau). \tag{32}$$

Proof: First, it is obvious that $\mathcal{P}_2^{\tau}$ is the solution of the standard Riccati (29) which is associated with the Kalman filtering of the system (1) and (13) with $i = 2$.

Second, note that for $\tau > t_1$, $\hat{\mathbf{x}}(\tau,1)$ is the projection of $\mathbf{x}(\tau)$ into the linear space

$$\mathcal{L}\left\{\mathbf{w}_2(s)|_{0\le s\le t_1};\quad \mathbf{w}_1(s)|_{0<s<\tau}\right\}$$

and can be obtained by applying the projection formula as

$$\hat{\mathbf{x}}(\tau,1) = \int_0^{t_1} \langle \mathbf{x}(\tau), \mathbf{w}_2(s)\rangle Q_{\mathcal{V}_2}^{-1}(s)\mathbf{w}_2(s)ds + \int_{t_1}^{\tau} \langle \mathbf{x}(\tau), \mathbf{w}_1(s)\rangle Q_{\mathcal{V}_1}^{-1}(s)\mathbf{w}_1(s)ds. \tag{33}$$

By differentiating both sides of (33) with respect to τ, we have

$$\frac{d\hat{\mathbf{x}}(\tau,1)}{d\tau} = \Phi(\tau)\hat{\mathbf{x}}(\tau,1) + \langle \mathbf{x}(\tau), \mathbf{w}_1(\tau)\rangle Q_{\mathcal{V}_1}^{-1}(\tau)\mathbf{w}_1(\tau) = \Phi(\tau)\hat{\mathbf{x}}(\tau,1) + \mathcal{K}_1(\tau)\mathbf{w}_1(\tau) \tag{34}$$

where

$$\mathcal{K}_1(\tau) = \mathcal{P})_1(\tau)\mathcal{H}_1'(\tau)Q_{\mathcal{V}_1}^{-1}(\tau)$$

with initial condition $\hat{\mathbf{x}}(t_1,2)$. Therefore, $\Sigma_1(\tau) \triangleq \langle\hat{\mathbf{x}}(\tau,1),\ \hat{\mathbf{x}}(\tau,1)\rangle$ satisfies the equation

$$\frac{d\Sigma_1(\tau)}{d\tau} = \Phi(\tau)\Sigma_1(\tau) + \Sigma_1(\tau)\Phi'(\tau) + \mathcal{K}(\tau)_1 Q_{\mathcal{V}_1}(\tau)\left[\mathcal{K}_1(\tau)\right]'. \tag{35}$$

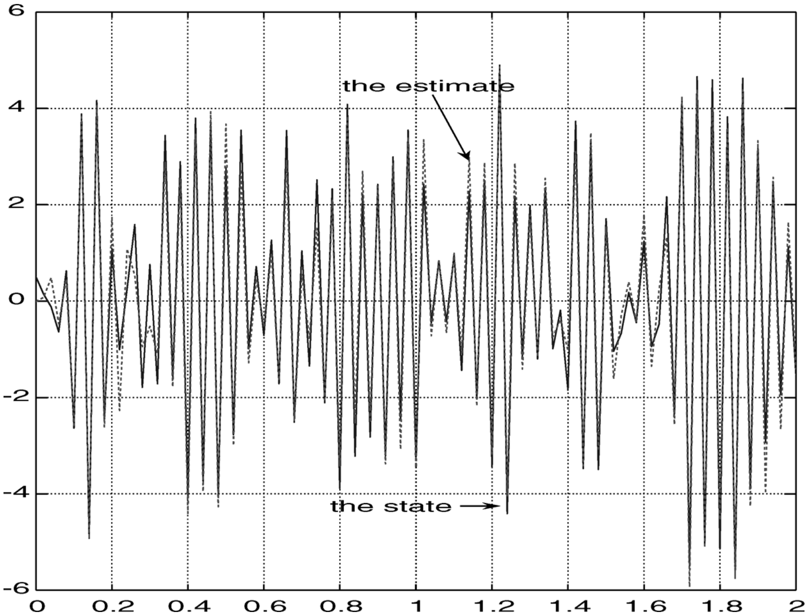

Fig. 2. True state $\hat{\mathbf{x}}_2(t|t)$ and its estimation value with delay.

It is also clear that $\Pi(\tau) \triangleq \langle \mathbf{x}(\tau), \mathbf{x}(\tau)\rangle$ satisfies the linear differential equation

$$\frac{d\Pi(\tau)}{d\tau} = \Phi(\tau)\Pi(\tau) + \Pi(\tau)\Phi'(\tau) + \Gamma(\tau)Q_u(\tau)\Gamma'(\tau). \quad (36)$$

In view of the decomposition $\mathbf{x}(\tau) = \hat{\mathbf{x}}(\tau, 1) + \tilde{\mathbf{x}}(\tau, 1)$, we have that $\Pi(\tau) = \Sigma_1(\tau) + \mathcal{P}_1(\tau)$. Then, (31) follows, which completes the proof.

C. Optimal Estimate $\hat{\mathbf{x}}(t|t)$

In this subsection, we will calculate the optimal estimate $\mathbf{x}(t|t)$ based on the Riccati equation given in last subsection.

Theorem 3.2: Consider (1)–(3) and given time $t > d$, the optimal estimate $\hat{\mathbf{x}}(t|t)$ is given by

$$\hat{\mathbf{x}}(t|t) = \hat{\mathbf{x}}(t, 1) \quad (37)$$

where $\hat{\mathbf{x}}(t, 1)$ is computed through the following steps.

Step 1) Calculating $\hat{\mathbf{x}}(\tau, 2)$ for $\tau = t_1$

$$\frac{d\hat{\mathbf{x}}(\tau, 2)}{d\tau} = \Phi(\tau)\hat{\mathbf{x}}(\tau, 2) + \mathcal{K}_2(\tau)\left[\mathcal{Y}_2(\tau) - \mathcal{H}_2(\tau)\hat{\mathbf{x}}(\tau, 2)\right]$$
$$\hat{\mathbf{x}}(0, 2) = 0 \quad (38)$$

where $\mathcal{K}_2(\tau) = \mathcal{P}_2(\tau)\mathcal{H}_2'(\tau)Q_{\mathcal{V}_2}^{-1}(\tau)$ and $\mathcal{P}_2(\tau)$ is calculated by (29), i.e.,

$$\frac{d\mathcal{P}_2(\tau)}{d\tau} = \Phi(\tau)\mathcal{P}_2(\tau) + \mathcal{P}_2(\tau)\Phi'(\tau) - \mathcal{K}_2(\tau)Q_{\mathcal{V}_2}(\tau)\left[\mathcal{K}_2(\tau)\right]'$$
$$+\Gamma(\tau)Q_u(\tau)\Gamma'(\tau) \quad \mathcal{P}_2(0) = \Pi_0. \quad (39)$$

Step 2) Calculating $\hat{\mathbf{x}}(\tau, 1)$ for $t_1 < \tau \leq t$

$$\frac{d\hat{\mathbf{x}}(\tau, 1)}{d\tau} = \Phi(\tau)\hat{\mathbf{x}}(\tau, 1) + \mathcal{K}_1(\tau)\left[\mathcal{Y}_1(\tau) - \mathcal{H}_1(\tau)\hat{\mathbf{x}}(\tau, 1)\right]$$
$$\hat{\mathbf{x}}(t_1, 1) = \hat{\mathbf{x}}(t_1, 2) \quad (40)$$

where $\mathcal{K}_1(\tau) = \mathcal{P}_1(\tau)\mathcal{H}_1'(\tau)Q_{\mathcal{V}_1}^{-1}(\tau)$, and $\mathcal{P}_1(\tau)$ is calculated by (31), i.e.,

$$\frac{d\mathcal{P}_1(\tau)}{d\tau} = \Phi(\tau)\mathcal{P}_1(\tau) + \mathcal{P}_1(\tau)\Phi'(\tau) - \mathcal{K}_1(\tau)Q_{\mathcal{V}_1}(\tau)\left[\mathcal{K}_1(\tau)\right]'$$
$$+\Gamma(\tau)Q_u(\tau)\Gamma'(\tau) \quad \mathcal{P}_1(t_1) = \mathcal{P}_2(t_1). \quad (41)$$

Step 3) The estimator $\hat{\mathbf{x}}(t, 1)$ is computed from Step 2) for $\tau = t$.

Proof: (37) is obtained immediately from Definition 3.1. (39) and (41) follow directly from (29) and (31) respectively. From the standard Riccati equation, we can conclude (38). From (17) and (34), we can obtain (40).

Remark 3.1: When $0 \leq t \leq d$, the optimal estimator $\hat{\mathbf{x}}(t|t)$ is the projection of state $\mathbf{x}(t)$ onto the linear space generated by the observation of $\{\mathbf{y}(\tau),\ 0 \leq \tau \leq t\}$. Note the observation $\mathbf{y}(\tau)$ is from (2) and delay free, thus $\hat{\mathbf{x}}(t|t)$ is standard Kalman filter associated with system (1), (2) which can be computed by

$$\frac{d\hat{\mathbf{x}}(\tau, 1)}{d\tau} = \Phi(\tau)\hat{\mathbf{x}}(\tau, 1) + \mathcal{K}_1(\tau)\left[\mathbf{y}_1(\tau) - \mathcal{H}_1(\tau)\hat{\mathbf{x}}(\tau, 1)\right]$$
$$\hat{\mathbf{x}}(0, 1) = 0 \quad (42)$$

where $\mathcal{K}_1(\tau) = \mathcal{P}_1(\tau)\mathcal{H}_1'(\tau)Q_{\mathcal{V}_1}^{-1}(\tau)$, and $\mathcal{P}_1(\tau)$ is calculated by

$$\frac{d\mathcal{P}_1(\tau)}{d\tau} = \Phi(\tau)\mathcal{P}_1(\tau) + \mathcal{P}_1(\tau)\Phi'(\tau) - \mathcal{K}_1(\tau)Q_{\mathcal{V}_1}(\tau)\left[\mathcal{K}_1(\tau)\right]'$$
$$+\Gamma(\tau)Q_u(\tau)\Gamma'(\tau) \quad \mathcal{P}_1(0) = \Pi_0. \quad (43)$$

When $t > d$, a flow chart for calculating $\hat{\mathbf{x}}(t|t)$ is drawn in Fig. 4.

IV. Numerical Example

In the section, we illustrate the results obtained in previous section with a simple example. Consider the continuous-time model (1)–(3) with

$$\Phi(t) = \begin{bmatrix} -10 & 0 \\ 10 & -20 \end{bmatrix} \quad \Gamma(t) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$
$$H(t) = [1 \quad 1] \quad H_1(t) = [2 \quad 1] \quad (44)$$

then

$$\mathcal{H}_1(t) = [1 \quad 1] \quad \mathcal{H}_2(t) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}. \quad (45)$$

The initial state $\mathbf{x}(0) = \left[\begin{smallmatrix} 1 \\ 0.5 \end{smallmatrix}\right]$, $\hat{\mathbf{x}}(0) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and $P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In the simulation, $\mathbf{u}(t)$, $\mathbf{v}_0(t)$ and $\mathbf{v}_1(t)$ are assumed to be uncorrelated Gaussian noises with zero means and known covariance matrices $Q_u(t) = 1$, $Q_{v_0}(t) = Q_{v_1}(t) = 1$, let sampling period $Ts = 0.02$ s, $d = 0.4$ s.

The simulation results are drawn in Figs. 1–3. It can be observed from the simulation results that the proposed method produces very good performance, so the technique proposed in this note is efficient.

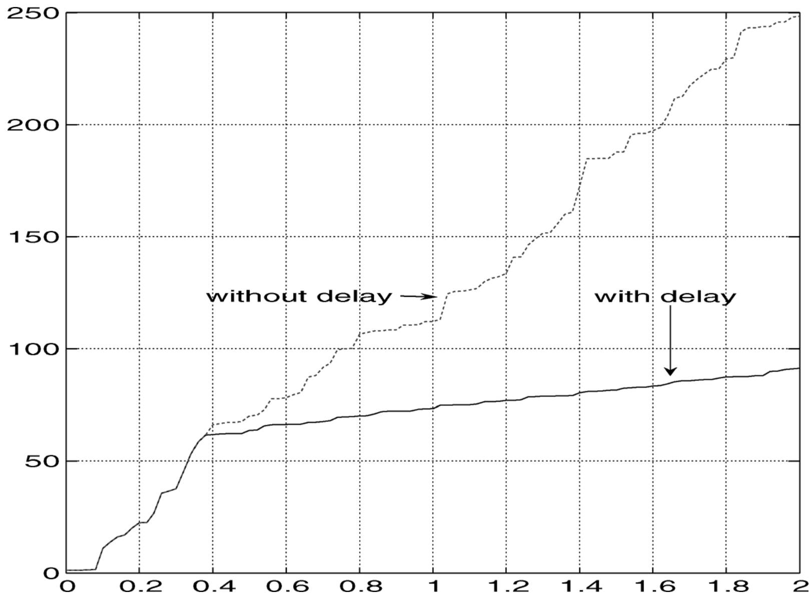

Fig. 3. Sum of the variance with delay and without delay.

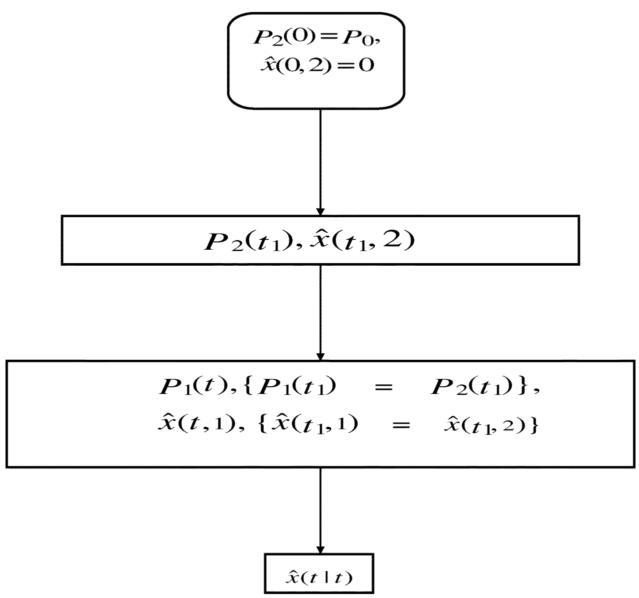

Fig. 4. Flow chart for calculating $\hat{\mathbf{x}}(t|t)$ with $t > d$.

V. Conclusion

The note studied the Kalman filtering for continuous-time systems with delayed measurements. The optimal estimate is derived by developing a new tool called *reorganized innovation approach*. It consists of two different Kalman filters that have the same dimension as the original system. Comparing with the conventional approach via solving partial differential equation (PDE), a significant advantage of new approach is that it provides a closed-form solution. Simulation results showed that the new approach is efficient. Moreover, we believe that the presented results give an useful benchmark for dealing with time delay problems such as controller and filter design in H_2 or H_∞ performance [8], [9], [13].

It should be pointed out that the presented results in this note are limited to the systems with only current and one channel delayed measurements. This is in comparison with some recent publications as in [1] and [2], where the more general cases for the systems with both measurement delays and state delays were considered.

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