Journal of Control Theory and Applications 1 (2006) 53–61

Stabilization of a class of switched nonlinear systems

Daizhan CHENG [1], Gang FENG [2], Zairong XI [1]

(1. Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, China;

2. Department of Manufacturing Engineering and Engineering Management, City University of HongKong, HongKong)

Abstract: The stabilization of a class of switched nonlinear systems is investigated in the paper. The systems concerned are of (generalized) switched Byrnes-Isidori canonical form, which has all switched models in (generalized) Byrnes-Isidori canonical form. First, a stability result of switched systems is obtained. Then it is used to solve the stabilization problem of the switched nonlinear control systems. In addition, necessary and sufficient conditions are obtained for a switched affine nonlinear system to be feedback equivalent to (generalized) switched Byrnes-Isidori canonical systems are presented. Finally, as an application the stability of switched lorenz systems is investigated.

Keywords: Switched nonlinear systems; Stabilization; Canonical form; Common Lyapunov function

1 Introduction

In recent years, the study of switched systems has received more and more attention. Many engineering systems, such as robot manipulators [1], traffic management [2], power systems [3,4], etc. are essentially switched systems.

Stabilization is a fundamental task for control design of switched systems [5]. The stabilization of switched linear systems has been investigated for a long time. Many advanced tools have been used in the study. For example, in [6] Lie-algebra structure was used to reveal a sufficient stability criteria for switched linear systems. Multiple Lyapunov functions was used in [7]. Quadratic stability by using quadratic Lyapunov functions (QLF) has been shown to be an effective way to solve the problem of stability and stabilization of switched linear systems [8,9]. However, it is known that the existence of a common quadratic Lyapunov function is not necessary for the stability of switched linear systems [10]. A recent result provides a necessary and sufficient condition for the existence of a common quadratic Lyapunov function for a set of stable matrices [11].

Recently, the stabilization of switched nonlinear systems has also been investigated [12,13]. The switched technique is implemented for the stabilization of some typical kinds of nonlinear systems [14,15].

A general switched affine nonlinear control system concerned is described as

$$\begin{aligned} &\text{a)}\quad \dot{\xi} = f^{\sigma(t)}(\xi) + g^{\sigma(t)}(\xi)u, \xi(t) \in \mathbb{R}^n,\ u \in \mathbb{R}^m;\\ &\text{b)}\quad y_i = h_i(\xi), i = 1, \cdots, m, \end{aligned} \tag{1}$$

where the switching law $\sigma(t) : [0,\infty) \to \Lambda$ is a right-continuous piecewise constant mapping, and $\Lambda = \{1, 2, \cdots, N\}$ for some integer $N \geqslant 2$.

Throughout the paper, we assume that the switching law can be arbitrary.

The systems considered in this paper are mainly of the following two particular forms. The first one is

$$\begin{aligned} &\text{a)}\quad \begin{cases} \dot{x} = A^{\sigma(t)}x + B^{\sigma(t)}u,\ x \in \mathbb{R}^{n-l},\ u \in \mathbb{R}^m,\\ \dot{z} = p^{\sigma(t)}(\xi),\ z \in \mathbb{R}^l; \end{cases}\\ &\text{b)}\quad y = Cx, \end{aligned} \tag{2}$$

where $\xi = (x, z)$, and (A^λ, B^λ), $\lambda \in \Lambda$ are completely

Received 30 September 2005; revised 9 December 2005.

This work is partly supported by the National Natural Science Foundation of China (No. 60274010, 60221301, 60334040, 60228003).

controllable pairs. The second one is

$$\text{a)}\quad \begin{cases} \dot{x} = f_1^{\sigma(t)}(x,z) + g_1^{\sigma(t)}(x,z)u,\ x \in \mathbb{R}^{n-l}, \\ \dot{z} = p^{\sigma(t)}(\xi),\ z \in \mathbb{R}^l, \end{cases}$$
$$\text{b)}\quad y = Cx, \tag{3}$$

where $\xi = (x,z)$, $f_1^\lambda(0,z) = 0$, and

$$A^\lambda = \left.\frac{\partial f_1^\lambda}{\partial x}\right|_{(0,0)}, \quad B^\lambda = g_1^\lambda(0,0) \tag{4}$$

are completely controllable pairs.

We call (2) switched Byrnes-Isidori canonical form (SBICF) because each switching model is in Byrnes-Isidori canonical form (BICF) (up to a linear transformation), and (3) generalized switched Byrnes-Isidori canonical form (GSBICF). Since in state feedback stabilization problem the outputs are less involved, for convenience, sometimes we also call (2(a)) SBICF and (3 (a)) GSBICF.

Note that a SBICF has not only all switched models in BICF, but also common linearly controllable sub-space for all switching models.

The purpose of this paper is to investigate the stabilization of switched nonlinear systems in SBICF (GSBICF). We will also consider when system (1) can be converted to (2) or (3) via state feedback, which is also said to be feedback equivalent to SBICF (GSBICF). The main tool for stabilization is the Lyapunov function with homogeneous derivatives (LFHD) proposed in [16].

The rest of this paper is organized as follows: section 2 provides a slightly improved stability result via LFHD. Section 3 proves a theorem on stability, which is a key result for the following stabilization argument. Stabilization problem is considered in section 4. Section 5 considers the transformation of general switched nonlinear control systems into SBICF (GSBICF). Section 6 is some concluding remarks.

2 Lyapunov function with homogeneous derivatives

Center manifold theory has been proven as a powerful tool for investigating stability and stabilization of nonlinear systems [16~18]. As a basic tool for a systematic center manifold approach to stabilization problems, the Lyapunov function with homogeneous derivatives (LFHD) has been proposed in [16]. This section restate it in a more general form.

Consider a nonlinear system

$$\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n. \tag{5}$$

Denote by $f_i(x)$, $i = 1, \cdots, n$, the components of $f(x)$. Then, we have the following generalized definition of approximate stability.

Definition 1 System (5) is said to be approximately stable at the origin with approximate degrees $(d_1, \cdots, d_n)$ if

$$\dot{x}_i = f_i(x) + O(\|x\|^{d_i+1}), \quad i = 1, \cdots, n \tag{6}$$

is asymptotically stable at the origin.

Now consider system (5) again. We split the components of $f(x)$ as

$$f_i(x) = q_i(x) + e_i(x), \quad i = 1, \cdots, n,$$

where $q_i(x)$ is a polynomial of degree $\deg(q_i(x)) = d_i$ and $e_i(x) = O(\|x\|^{d_i+1})$. Then we construct an approximated system as

$$\dot{x} = q(x), \quad x \in \mathbb{R}^n, \tag{7}$$

where $q(x) = (q_1(x), \cdots, q_n(x))^{\mathrm{T}}$. A positive definite polynomial $V(x) > 0$ is said to be a LFHD of (7) if

$$\deg\left(\frac{\partial V(x)}{\partial x_i} q_i(x)\right) = d, \quad \forall i.$$

Using above notations, we have the following result:

Theorem 1 The origin of (5) is approximately stable with approximate degrees $(d_1, \cdots, d_n)$, if there exists a LFHD $V(x) > 0$ ($x \in U \backslash \{0\}$), such that

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} q_i(x) < 0, \quad x \in U \backslash \{0\},$$

where $U \subset \mathbb{R}^n$ is a neighborhood of the origin. Particularly, system (5) is asymptotically stable at the origin.

Remark 1 Theorem 1 is a little bit more general than its original version in [16], where $q_i(x)$ is assumed to be the lowest degree terms of $f_i(x)$. But a straightforward verification shows that the proof in [16] remains available. We need this version in the sequel.

We give a simple example to describe the application of Theorem 1. Note that the corresponding theorem in [16] is not applicable to this example.

Example 1 Consider the following system

$$\begin{cases} \dot{x}_1 = -x_2^3 - x_1^5 + x_2^3(1-\cos x_1) \quad (:= f_1(x)), \\ \dot{x}_2 = x_1 - x_2^3 + x_1^2 \sin x_2 \quad (:= f_2(x)). \end{cases} \tag{8}$$

We construct its approximated system as

$$\begin{cases} \dot{x}_1 = -x_2^3 - x_1^5 + 0.5x_2^3x_1^2 \quad (:= g_1(x)), \\ \dot{x}_2 = x_1 - x_2^3 + x_1^2 x_2 \quad (:= g_2(x)). \end{cases} \tag{9}$$

Then

$$d_1 = \deg(g_1(x)) = 5,\ d_2 = \deg(g_2(x)) = 3.$$

Using Taylor series expansion, we have

$$e_1 = x_2^3(1-\cos x_1 - 0.5x_1^2) = O(\|x\|^7) \subset O(\|x\|^6);$$

$$e_2 = x_1^2(\sin x_2 - x_2) = O(\|x\|^5) \subset O(\|x\|^4).$$

Now we define a candidate of LFHD as

$$V(x) = 2x_1^2 + x_2^4 > 0, \quad (x \neq 0).$$

Using the following inequality [16]

$$a^p b^q \leqslant \frac{p}{p+q}|a|^{p+q} + \frac{q}{p+q}|b|^{p+q}, \\ a, b \in \mathbb{R},\ p > 0, q > 0, \tag{10}$$

we have

$$\begin{aligned} \dot{V}|_{(9)} &= -4x_1^6 + 2x_1^3x_2^3 - 4x_2^6 + 4x_1^2x_2^4 \\ &\leqslant -4x_1^6 + x_1^6 + x_2^6 - 4x_2^6 + 4\left(\frac{2}{6}x_1^6 + \frac{4}{6}x_2^6\right) \\ &= -\frac{5}{3}x_1^6 - \frac{1}{3}x_2^6 < 0, \quad (x \neq 0). \end{aligned}$$

According to Theorem 1, system (8) is approximately stable at the origin with approximate degrees (5,3). Particularly, it is asymptotically stable at the origin.

3 Stability

In this section we consider a class of switched affine nonlinear systems. The systems have the following two-block structure:

$$\begin{cases} \dot{x} = [A^{\sigma(t)} + H^{\sigma(t)}(\xi)]x,\ x \in \mathbb{R}^{n-l}, \\ \dot{z} = p^{\sigma(t)}(\xi),\ z \in \mathbb{R}^l, \end{cases} \tag{11}$$

where $\xi = (x, z)$, the switching law $\sigma(t) : [0, \infty) \to \Lambda$ is a right-continuous piecewise constant mapping, $\Lambda = \{1, 2, \cdots, N\}$ for some integer $N \geqslant 2$, and $H^\lambda(\xi)$ and $p^\lambda(\xi)$ $\lambda = 1, \cdots, N$, are C^∞ vector functions.

The following assumption is made to emphasize that $H^\lambda(\xi)$ is a higher degree term.

Assumption 1 $H^\lambda(\xi) = O(\|\xi\|)$, $\lambda \in \Lambda$.

Split $p^\lambda(0, z)$ as

$$p^\lambda(0, z) = q^\lambda(z) + \epsilon^\lambda(z), \quad \lambda \in \Lambda,$$

$\deg(q_i^\lambda(z)) = d_i^\lambda$ and $e_i^\lambda(z) = O(\|z\|^{d_i^\lambda+1})$. Using it, we construct an approximated system as

$$\dot{z}_i = q_i^\lambda(z), \quad i = 1, \cdots, l;\ \lambda \in \Lambda. \tag{12}$$

Then, we have our main stability result:

Theorem 2 System (11) with assumption 1 is asymptotically stable at the origin, if there exists a neighborhood, U, of $\xi = 0$ such that

i) A^λ, $\lambda \in \Lambda$ share a common QLF. That is, there exists a positive definite matrix, $P > 0$, such that

$$PA^\lambda + (A^\lambda)^{\mathrm{T}}P := Q^\lambda < 0, \quad \lambda \in \Lambda. \tag{13}$$

ii) there exist a common LFHD, $V(z) > 0$ for (12) and a set of integers $d^\lambda > 0$, $\lambda \in \Lambda$ such that

$$\deg\left(\frac{\partial V}{\partial z}q^\lambda(z)\right) = d^\lambda, \quad \forall \lambda \in \Lambda, \quad z \in U, \tag{14}$$

and

$$\frac{\partial V}{\partial z}q^\lambda(z) < 0 \quad \forall \lambda \in \Lambda, \quad z \in U; \tag{15}$$

iii) for each component of $p^\lambda(x, z)$, $\lambda \in \Lambda$ the lowest degree (LD) of its components satisfies

$$LD\left(p_j^\lambda(x, z)\right) \geqslant e_j^\lambda - \frac{e^\lambda}{2} + 1, \tag{16}$$

where $j = 1, \cdots, l,\ z \in U$.

Proof Choose a candidate of Lyapunov function as

$$L(\xi) = ax^{\mathrm{T}}Px + V(z), \quad a > 0,$$

where $a > 0$ is an adjustable parameter to be determined later. It suffices to show that the derivative of L is negative along trajectories of all switching models. For any $\lambda \in \Lambda$, we have

$$\dot{L}|_{(11)} = ax^{\mathrm{T}}\left(Q^\lambda\right)x + 2ax^{\mathrm{T}}PH^\lambda(\xi)x + \sum_{j=1}^{d}\frac{\partial V}{\partial z_j}p_j^\lambda(x, z),$$

where $Q^\lambda = PA^\lambda + (A^\lambda)^{\mathrm{T}}P < 0$.

Using Taylor series expansion on $p_j^\lambda(x, z)$ with respect to x, we have

$$\begin{aligned} p_j^\lambda(x, z) &= \alpha_j^\lambda(0, z) + \beta_j^\lambda(0, z)x + \sum_{i=2}^{\infty}\frac{1}{i!}\gamma_{ji}^\lambda(0, z)x^i \\ &= q_j^\lambda(z) + \epsilon_j^\lambda(z) + \beta_j^\lambda(0, z)x + r_j^\lambda(x, z)x^2. \end{aligned} \tag{17}$$

Certain explanation is necessary for equation (17). First, $x^k = \underbrace{x\otimes\cdots\otimes x}_{k}$, where $\otimes$ is the tensor product of matrices. Secondly, for the second identity, using Taylor series expansion again on $\alpha_j^\lambda(0,z)$, we have

$$\alpha_j^\lambda(0,z) = q_j^\lambda(z) + \epsilon_j^\lambda(z),$$

where $\deg(q_j^\lambda(z)) = d_j^\lambda$ and $\epsilon_j^\lambda(z)$ is the remainder with $\deg(\epsilon^j(z)) > d_j^\lambda$. Finally, $r_j^\lambda(x,z)x^2$ is the sum of terms of degree of x greater than or equal to 2.

Denote

$$\Phi_0(z) := \sum_{j=1}^{l}\frac{\partial V}{\partial z_j}q_j^\lambda(z).$$

According to (15), it is a negative definite polynomial of degree d^λ. Define

$$\Phi_1(z) := \sum_{j=1}^{l}\frac{\partial V}{\partial z_j}\epsilon_j^\lambda(z).$$

By definition

$$\Phi_1(z) = O(\|z\|^{d^\lambda+1}). \tag{18}$$

Define

$$\Phi_2(z) := \sum_{j=1}^{l}\frac{\partial V}{\partial z_j}\beta_j^\lambda(0,z).$$

Note that $\deg\left(\dfrac{\partial V}{\partial z_j}\right) = \mathrm{e}^\lambda - \mathrm{e}_j^\lambda$, then by condition (8), we have

$$\begin{aligned}\Phi_2(z) &= \sum_{j=1}^{l} O\left(\|z\|^{d^\lambda-d_j^\lambda}\right)O\left(\|z\|^{d_j^\lambda-d^\lambda/2}\right)\\ &= O(\|z\|^{d^\lambda/2}).\end{aligned} \tag{19}$$

Define

$$\Phi_3(x,z) := \sum_{j=1}^{l}\frac{\partial V}{\partial z_j}r_j^\lambda(x,z) = O(\|z\|), \tag{20}$$

which is from the fact that $\dfrac{\partial V}{\partial z_j} = O(\|z\|)$. Finally, we define

$$\Phi_4(x,z) := 2x^{\mathrm T}PH^\lambda(\xi) = x^{\mathrm T}PxO(\|(x,z)\|). \tag{21}$$

The estimation of (21) is from the assumption 1.

Summarizing the above we have

$$\begin{aligned}\dot L|_{(11)} &= ax^{\mathrm T}Q^\lambda x + \Phi_0^\lambda(z) + \Phi_1^\lambda(z)\\ &\quad + \Phi_2^\lambda(z)x + \Phi_3^\lambda(x,z)x^2 + \Phi_4^\lambda(x,z).\end{aligned} \tag{22}$$

Note that $[\Phi_2^\lambda(z)]^{\mathrm T} \in \mathbb{R}^{n-l}$ and $[\Phi_3^\lambda(x,z)]^{\mathrm T} \in \mathbb{R}^{2(n-l)}$ are two row vectors.

Then we have to show that there is a neighborhood U of $(0,0)$, such that (22) is negative definite over U. According to (18), (20), and (21), we do not need to worry about $\Phi_1^\lambda(z)$, $\Phi_3^\lambda(x,z)x^2$ and $\Phi_4^\lambda(x,z)$. Because they are higher order infinitesimals of $\Phi_0(z)$ and/or $x^{\mathrm T}Qx$ respectively as $(x,z)\to 0$.

Using Schwartz's inequality, we have for any $E_\lambda > 0$

$$\left|\Phi_2^\lambda(z)x\right| \leqslant \frac{1}{E_\lambda}\|x\|^2 + E_\lambda\|\Phi_2(z)\|^2. \tag{23}$$

Choosing $E_\lambda > 0$ to be small enough and according to the right hand side of (19), the second term in (23) can be dominated by $\Phi_0(z)$. Precisely, taking $\Phi_0(z)$, $\Phi_1(z)$, and $\Phi_2(z)$ into consideration simultaneously, we have $\Phi_0(z)/2 + \Phi_1(z) < 0$ by choosing small enough U_λ, and $\Phi_0(z)/2 + E_\lambda\|\Phi_2(z)\|^2 < 0$ by choosing small enough E_λ. We can also have both $\dfrac{a_\lambda}{3}x^{\mathrm T}Q^\lambda x + \Phi_3^\lambda(x,z)x^2 < 0$ and $\dfrac{a_\lambda}{3}x^{\mathrm T}Q^\lambda x + \dfrac{1}{E_\lambda}\|x\|^2 < 0$ by taking large enough a_λ, and $\dfrac{a_\lambda}{3}x^{\mathrm T}Q^\lambda x + 2a_\lambda\Phi_4^\lambda(z) < 0$ by shrinking U_λ if necessary. It then follows that for a fixed $\sigma(t) = \lambda$ and setting $a = a_\lambda$, we have

$$\dot L|_{(11)}(x,z) < 0,\quad 0 \neq (x,z) \in U_\lambda. \tag{24}$$

Finally, choosing

$$E = \min\nolimits_{\lambda\in\Lambda}(E_\lambda),\quad a = \max\nolimits_{\lambda\in\Lambda}(a_\lambda),\quad U = \bigcap_{\lambda\in\Lambda}(U_\lambda),$$

then (24) is true for all $\lambda \in \Lambda$, which implies that system (11) is asymptotically stable.

4 Stabilization

This section considers the stabilization of SBICF system (2) and GSBICF system (3). We give the following result for GSBICF systems. It is obviously applicable to SBICF systems.

Using the same notations as in Theorem 2, a straightforward computation leads to the following result.

Theorem 3 System (3) is locally stabilizable, if there exists a neighborhood, U, of $\xi = 0$ such that

i) (A^λ, B^λ), $\lambda \in \Lambda$ are simultaneously quadratically stabilizable. That is, there exist state feedbacks

$$u^\lambda = K^\lambda x,\quad \lambda \in \Lambda, \tag{25}$$

and a positive definite matrix, $P > 0$, such that $\tilde A^\lambda = A^\lambda + B^\lambda K^\lambda$, $\lambda \in \Lambda$ satisfy

$$P\tilde A^\lambda + (\tilde A^\lambda)^{\mathrm T}P := Q^\lambda < 0,\quad \lambda \in \Lambda. \tag{26}$$

ii) there exists a common LFHD, $V(z) > 0$, for the approximate switched system (12) and a set of integers

$d^\lambda > 0$, $\lambda \in \Lambda$ such that

$$\deg\left(\frac{\partial V}{\partial z} q^\lambda(z)\right) = d^\lambda, \quad \forall \lambda \in \Lambda, \quad z \in U, \tag{27}$$

and

$$\frac{\partial V}{\partial z} q^\lambda(z) < 0 \quad \forall \lambda \in \Lambda, \quad z \in U; \tag{28}$$

iii) for each component of $p^\lambda(x,z)$, $\lambda \in \Lambda$ the lowest degrees (LD) of its components satisfy

$$LD\left(p_j^\lambda(x,z)\right) \geqslant e_j^\lambda - \frac{e^\lambda}{2} + 1, \tag{29}$$

where $j = 1, \cdots, d$, $z \in U$. Moreover, u^λ, defined in (25) stabilizes the overall system.

Proof Consider (3) a). Since $f_1^\lambda(0,z) = 0$, using Taylor series expansion on $f_1^\lambda(x,z)$, we have

$$f_1^\lambda(x,z) = A^\lambda x + \alpha^\lambda(\xi) x^2.$$

Similarly,

$$g_1^\lambda(x,z) = B^\lambda + \beta^\lambda(\xi)\xi.$$

Plugging them and $u = K^\lambda x$ into (3) a) yields a closed-loop form, which is exactly the same as (11). The conclusion follows from Theorem 2.

Example 2 Consider the following system

$$\begin{cases} \dot{x} = A^{\sigma(t)} x + b^{\sigma(t)} u + x^{\mathrm{T}} \eta^{\sigma(t)}(x,z) z, \\ \dot{z} = p^{\sigma(t)}(x,z). \end{cases} \tag{30}$$

$\Lambda = \{1, 2\}$, $\eta^\lambda(x,z)$, $\lambda \in \Lambda$ are any 2×2 smooth matrix, and

$$A_1 = \begin{bmatrix} -3 & -3 \\ 0 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix};$$

$$b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, p_1 = \begin{bmatrix} z_1^2 \tan(z_2^3 - z_1^3) + x_1 z_2^3 \\ \tan(z_1^3 - z_2^3) + x_2 z_1^2 \end{bmatrix},$$

$$b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, p_2 = \begin{bmatrix} \ln(1 - z_1^3) + (x_1 + x_2) z_1^2 \\ 1 - 2\sin(z_2) - \exp(z_1) \end{bmatrix}.$$

Note that the switching models of system (30) are in GSBICF. We use Theorem 3 to show the system (30) is stabilizable. For the linear sub-system (we refer to [19] for systematic treatment), taking

$$u^1 = K_1 x = \begin{bmatrix} 1 & 2 \end{bmatrix} x, \; u^2 = K_2 x = \begin{bmatrix} -1 & 1 \end{bmatrix} x \tag{31}$$

leads to the feedback linear systems with matrices as

$$\tilde{A}_1 = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix}.$$

Then, $\tilde{A}_1$ and $\tilde{A}_2$ share a common quadratic Lyapunov function $x^{\mathrm{T}} P x$ with

$$P = \begin{bmatrix} 7 & -4 \\ -4 & 3 \end{bmatrix}.$$

Next, we consider the zero dynamics (with respect to $y = x$). For $\lambda = 1$, we have

$$\begin{cases} \dot{z}_1 = z_1^2 \tan(z_2^3 - z_1^3), \\ \dot{z}_2 = \tan(z_1^3 - z_2^3). \end{cases} \tag{32}$$

For $\lambda = 2$, we have

$$\begin{cases} \dot{z}_1 = \ln(1 - z_1^3), \\ \dot{z}_2 = 1 - 2\sin(z_2) - \exp(z_1). \end{cases} \tag{33}$$

Their approximated systems are respectively

$$\begin{cases} \dot{z}_1 = -z_1^5 + z_1^2 z_2^3, \\ \dot{z}_2 = z_1^3 - z_2^3, \end{cases} \tag{34}$$

and

$$\begin{cases} \dot{z}_1 = -z_1^3, \\ \dot{z}_2 = -2z_2 - z_1. \end{cases} \tag{35}$$

It is easy to verify that $V(z) = z_1^2 + z_2^4$ is a common LFHD. Because

$$\dot{V}|_{(34)} = -2z_1^6 - 2z_1^3 z_2^3 - 4z_2^6 < 0,$$

and

$$\begin{aligned} \dot{V}|_{(35)} &= -2z_1^4 - 8z_2^4 - 4z_1 z_2^3 \\ &\leqslant -2z_1^4 - 8z_2^4 + z_1^4 + 3z_2^4 < 0. \end{aligned}$$

The last two inequalities are obtained by using the inequality (10).

Finally, we have for $\lambda = 1$:

$$d^1 = 6, \; LD(p_1^1(\xi)) = d_1^1 = 5, \; LD(p_2^1(\xi)) = d_2^1 = 3;$$

for $\lambda = 2$:

$$d^1 = 4, \; LD(p_1^2(\xi)) = d_1^2 = 3, \; LD(p_2^2(\xi)) = d_2^2 = 1.$$

It is easy to check that (29) is true. System (30) is stabilized by control (31).

5 On SBICF (GSBICF)

This section is mainly used to investigate when an affine switched system can be converted into SBICF (GSBICF) and hence the result obtained in previous section is applicable.

To begin with, we give an example to show that some mechanical systems can have this kinds of structure if some of its parts are flexible for different sizes.

Example 3 Consider a car with inverted pendulum of variable length l, switching among $\{l_1, \cdots, l_N\}$ (ref. Fig. 1).

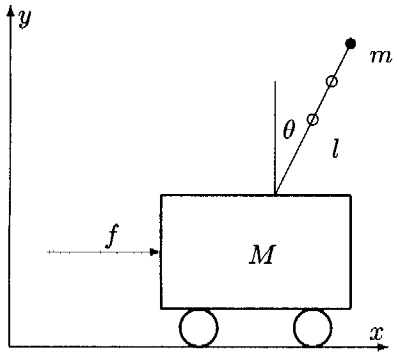


Fig. 1 Car with inverted pendulum.

The model can be expressed as [20]

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_3 = -z_4 z_5, \\ \dot{z}_4 = z_3 z_5, \\ \begin{bmatrix} \dot{z}_2 \\ \dot{z}_5 \end{bmatrix} = \begin{bmatrix} M+m & ml_{\sigma(t)}z_3 \\ ml_{\sigma(t)}z_3 & ml^2_{\sigma(t)} \end{bmatrix}^{-1} \begin{bmatrix} f + ml_{\sigma(t)}z_4 z_5^2 \\ mgl_{\sigma(t)}z_4 \end{bmatrix}, \end{cases} \tag{36}$$

where $z_1 = x$, $z_2 = \dot{x}$, $z_3 = \cos\theta$, $z_4 = \sin\theta$, $z_5 = \dot{\theta}$. Define

$$D = \left[(M+m) - mz_2^2\right] ml_{\sigma(t)},$$

$$\tilde{z}_5 = z_5 + (1/l_{\sigma(t)}) z_2 z_3,$$

and assume that the output is $y = z_1$, then (36) becomes

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = \dfrac{m}{D}\left[z_4(\tilde{z}_5 - z_2 z_3/l_{\sigma(t)})^2 + l_{\sigma(t)} f\right], \\ \dot{z}_3 = -z_4(\tilde{z}_5 - z_2 z_3/l_{\sigma(t)}), \\ \dot{z}_4 = z_3(\tilde{z}_5 - z_2 z_3/l_{\sigma(t)}), \\ \dot{\tilde{z}}_5 = \dfrac{1}{D} mgz_4 + \dfrac{1}{Dl_{\sigma(t)}} mz_4(\tilde{z}_5 \\ \qquad - z_2 z_3/l_{\sigma(t)})^2 z_3 - z_2 z_4(\tilde{z}_5 - z_2 z_3/l_{\sigma(t)}). \end{cases} \tag{37}$$

Obviously, (37) is of GSBICF.

Systems (2) and (3) seem to be a "block-wise switched" system. This kind of switched systems come from many practical systems. Say, switching with "building block" is fundamental in biological evolution [21]. Block-wise switching happens also in many power systems [22]. These are some motivations for studying (2) and (3).

Consider system (1). It is well known that under certain regular conditions [18] (1) can be expressed into B-I normal form as

$$\begin{cases} \dot{x}^\lambda(t) = A^\lambda x^\lambda + B^\lambda u, & x^\lambda(t) \in \mathbb{R}^s,\ u(t) \in \mathbb{R}^m, \\ \dot{z}^\lambda = p^\lambda(x^\lambda, z^\lambda), & z \in \mathbb{R}^{n-s}, \\ y = C^\lambda x^\lambda, & y \in \mathbb{R}^m, \quad \lambda \in \Lambda, \end{cases} \tag{38}$$

where (A^λ, B^λ), $\lambda \in \Lambda$, are controllable pairs.

We consider when the state equation of system (38) can be converted into the form of (2). Following standard procedure, for each model λ we denote the relative degree vector as

$$\rho^\lambda = \left(\rho_1^\lambda, \cdots, \rho_m^\lambda\right), \quad \lambda \in \Lambda.$$

Denote a set of distributions as

$$G_\lambda = \mathrm{Span}\{g_1^\lambda, \cdots, g_m^\lambda\}, \quad \lambda \in \Lambda.$$

Then, we define the decoupling matrices as

$$D_\lambda = \begin{bmatrix} L_{g_1^\lambda} L_{f^\lambda}^{\rho_1^\lambda} h_1 & \cdots & L_{g_m^\lambda} L_{f^\lambda}^{\rho_1^\lambda} h_1 \\ \vdots & & \vdots \\ L_{g_1^\lambda} L_{f^\lambda}^{\rho_m^\lambda} h_m & \cdots & L_{g_m^\lambda} L_{f^\lambda}^{\rho_m^\lambda} h_m \end{bmatrix}, \ \lambda \in \Lambda.$$

Finally, the sets of one-forms for $\lambda \in \Lambda$ can be produced as

$$\Omega_\lambda := \left\{dh_1, \cdot, dL_{f^\lambda}^{\rho_1^\lambda - 1} h_1, \cdots, dh_m, \cdot, dL_{f^\lambda}^{\rho_m^\lambda - 1} h_m\right\}.$$

We denote the module of Ω_λ over $\mathbb{R}$ (that is, the vector space spanned by Ω_λ over $\mathbb{R}$) by

$$M_\lambda := \mathrm{Mod}_{\mathbb{R}}\{\Omega_\lambda\}, \quad \lambda \in \Lambda.$$

The following result tells when system (1) has the SBICF.

Proposition 1 System (1) with regular feedbacks

$$u = \alpha^\lambda(\xi) + \beta^\lambda(\xi) v,$$

(where $\beta^\lambda(\xi)$ is locally invertible), has a SBICF, if i) $\rho_1^\lambda + \cdots + \rho_m^\lambda = s$, $\forall \lambda \in \Lambda$; ii) G_λ, $\lambda \in \Lambda$ are non-singular and involutive; iii) D_λ, $\lambda \in \Lambda$ are non-singular; and (iv) M_λ, $\lambda \in \Lambda$ are the same.

Proof i)~iii) are standard for B-I normal form using classical coordinate transformation based on Ω_λ [18].

Therefore, we can obtain (36) with the same size of linearized part. Finally, it is obvious that iv) is equivalent to that the linear parts of different models are linearly related. We, therefore, can choose any x^λ as a universal x. Then any z^λ can be chosen as universal z to get (2).

Next, we consider (3). In fact, we may be only interested in (3) a), as in many stabilization problems. Then we have the following easily proved result.

Proposition 2 System (1) a) is static state feedback equivalent to (3) a), if all the linear approximations (A^λ, B^λ), as defined in (4), are controllable, and

$$M_\lambda := \mathrm{Mod}_{\mathbb{R}}\{\Omega_\lambda\}, \quad \lambda \in \Lambda.$$

are the same, where

$$\Omega_\lambda = \{B^\lambda, A^\lambda B^\lambda, \cdots, (A^\lambda)^{l-1} B^\lambda\}.$$

6 Application to switched Lorenz system

Consider the Lorenz system [23]

$$\begin{cases} \dot{x} = \alpha(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = -\beta z + xy, \end{cases} \tag{39}$$

where $\alpha > 0$, $\beta > 0$, $\rho > 0$. Set $\rho = 1$, then the eigenvalues of the Jacobian matrix at the origin are 0, $-\alpha - 1$, and $-\beta$. It was shown in [24] that a linear switching can generate chaos. In this section, we want to ask whether arbitrary switching quadratic terms affects the stability of (39). Switching quadratic terms leads to the following model:

$$\begin{cases} \dot{x} = \alpha(y - x), \\ \dot{y} = x - y - f_{\sigma(t)}, \\ \dot{z} = -\beta z + g_{\sigma(t)}, \end{cases} \tag{40}$$

where $\sigma \in \Lambda = \{1, 2\}$, $f_1 = -xz$, $g_1 = xy$, $f_2 = g_1$, $g_2 = f_1$. Using coordinate transformation

$$\begin{bmatrix} u \\ v \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\alpha} & \frac{\alpha}{1+\alpha} & 0 \\ \frac{1}{1+\alpha} & \frac{-1}{1+\alpha} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and then

$$w = z - \frac{u^2}{\beta},$$

the following switched system can be obtained:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(1+\alpha) & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} -\frac{\alpha}{1+\alpha} f_{\sigma(t)} \\ -\frac{1}{1+\alpha} f_{\sigma(t)} \\ -u^2 + g_{\sigma(t)} - \frac{2u\alpha}{\beta(1+\alpha)} f_{\sigma(t)} \end{bmatrix}, \tag{41}$$

where

$$f_1 = g_2 = (u + \alpha v)(w + \frac{u^2}{\beta});$$
$$f_2 = g_1 = (u + \alpha v)(u - v).$$

Using Theorem 2, we have

$$q^1(u) = -u^3, \quad q^2(u) = -u^2.$$

Choose $V = u^{2k}$, one sees easily that as $k > 0$ large enough condition (16) is obviously satisfied. Condition (14) is trivial. To make (15) true we need $u \geqslant 0$. Note that $\{(u, v, w) \in \mathbb{R}^3 \mid u = 0\}$ is an invariant set of the system (41). Then $\{(u, v, w) \in \mathbb{R}^3 \mid u \geqslant 0\}$ is also an invariant set.

We conclude that if the initial value of the state of system (40) is in a certain neighborhood U of the origin and

$$\frac{1}{1+\alpha} x_0 + \frac{\alpha}{1+\alpha} y_0 \geqslant 0,$$

then the state converges to the origin. Hence no chaos will happen even under arbitrary switchings.

7 Conclusions

In this paper, the problem of stabilization has been studies for a class of switched nonlinear systems. The systems concerned are in SBICF or GSBICF. First, a stability result, based on common quadratic Lyapunov function (for linear part) and common LFHD (for zero dynamics), has obtained. It provides a tool for stabilizations of (general) SBICF. An example was provided to describe the stabilization procedure. Then we consider when a switched affine nonlinear system is transferable, via static state feedback, to SBICF and GSBICF. Finally, the result is applied to investigate the dynamic behavior of switched Lorenz systems.

Only systems in SBICF (GSBICF) were considered. Moreover, the zero dynamics of all switching models were assumed to be of minimum phase. These two are rigorous constrains. The stabilization problem of switched nonlinear systems both with non-comparable sub-space of linear part and with non-minimum phase zero dynamics remain for further study.

References

[1] D. Jeon, M. Tomizuka. Learning hybrid force and position control of robot manipulators[J]. *IEEE Trans. on Robotics and Automation*, 1993, 9(4): 423-431.

[2] P.P. Varaiya. Smart car on smart roads: problems of control[J]. *IEEE Trans. on Automatic Cortrol*, 1993, 38(2): 195-207.

[3] H. Sira-Ranirez. Nonlinear P-I controller design for switch mode dc-to dc power converters[J]. *IEEE Trans. Circuits and Systems*, 1991, 38(4): 410-417.

[4] S.M. Williams, R.G. Hoft. Adaptive frequency domain control of PM switched power line conditioner[J]. *IEEE Trans. Power Electronics*, 1991, 6(10): 665-670.

[5] D. Liberzon, A.S. Morse. Basic problems on stability and design of switched systems[J]. *IEEE Control Systems Magazine*, 1999, 19(5): 59-70.

[6] A.A. Agrachev, D. Liberzon. Lie-algebraic stability criteria for switched systems[J]. *SIAM J. of Control Optimization*, 2001, 40(1): 253-269.

[7] M.S. Branicky. Multiple Lyapunov functions and other analysis tools for switched hybrid systems[J]. *IEEE Trans. on Automatic Control*, 1998, 43(4): 475-482.

[8] R.N. Shortenm, K.S. Narendra. A sufficient condition for the existence of a common Lyapunov function for two second-order linear systems[C] // *Proc. 36th Conf. on Decision and Control*. New York: IEEE Press, 1997: 3521-3522.

[9] R.N. Shortenm, K.S. Narendra. Necessary and sufficient condition for the existence of a common quadratic Lyapunov function for M-stable linear second order systems[C] // *Proc. of the 2002 American Control Conf.*. Danvers, Massachusetts: American Automatic Control Council, 2002, 2: 1296-1301.

[10] W.P. Dayawansa, C.F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching[J]. *IEEE Trans. on Automatic Control*, 1999, 44(4): 751-760.

[11] D. Cheng, L. Guo, J. Huang. On quadratic Lyapunov functions[J]. *IEEE Trans. on Automatic Control*, 2003, 48(5): 885-890.

[12] J.L. Mancilla-Aguilar, R.A. Garcua. A converse Lyapunov Theorem for nonlinear switched systems[J]. *Systems & Control Letters*, 2000, 44(1): 67-71.

[13] Mancilla-Aguilar. A conditin for the stability of switched nonlinear systems[J]. *IEEE Trans. on Automatic Control*, 2000, 45(11): 2077-2079.

[14] J. Zhao, M.W. Spong. Hybrid control for global stabilization of the cart-pendulum system[J]. *Automatica*, 2001, 37(12): 1941-1951.

[15] W. Chen, D.J. Balance. On a switching control scheme for nonlinear systems with ill-defined relative degree [J]. *Systems & Control Letters*, 2002, 47(2): 159-166.

[16] D. Cheng, C. Martin. Stabilization of nonlinear systmes via designed center manifold[J]. *IEEE Trans. on Automatic Control*, 2001, 46(9):1372-1383.

[17] D. Aeyels. Stabilization of a class of nonlinear systems by a smooth feedback control[J]. *Systems & Control Letters*, 1985, 5(5): 289-294.

[18] A. Isidori. *Non-linear Control Systems*[M]. 3rd ed. Berlin: Springer-Verlag, 1995.

[19] D. Cheng. Stabilization of planar linear systems[J]. *Systems & Control Letters*, 2004, 51(2): 79-88.

[20] R. Lozano, I. Fantoni. Passivity based control of the inverted pendulum[C] // Huijberts, H.J.C., Nijmeijer, H., van der Schaft, A.J., Scherpen, J.M.A., Eds. *Proc. of Symposium on Nonlinear Control Systems*. Oxford: Elsevier Science, 1998: 145-150.

[21] J.H. Holland. *Hidden Order, How Adaptation Builds Complexity*[M]. New York: Addisom Wesley, 1995.

[22] G. Escobar, A.J. van der Schaft, R. Ortega. A Hamiltonian viewpoint in the modeling of switching power converters[J]. *Automatica*, 1999, 35(3): 445-452.

[23] J. Guckenheimer, P. Holmes. *Nonlinear Oscullations, Dynamical Systems, and Bifurcations of Vector Fields*[M]. New York: Springer-Verlag, 1983.

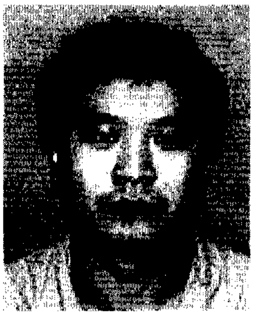
[24] J. Lu, T. Zhou, G. Chen, X. Yang. Generating chaos with a switching piecewise-linear controller[J]. *Chaos*, 2002, 12(2): 344-349.

[25] J. Carr. *Applications of Center Manifold Theory*[M]. New York: Springer-Verlag, 1981.

[26] D. Cheng, L. Zhang. Generalized normal form and stabilization of nonlinear systems[J]. *Int. J. of Control*, 2003, 76(2): 116-128.

Daizhan CHENG received Ph.D. degree from Washington University, MO, in 1985. Since 1990, he is a professor with the Institute of Systems Science. Chinese Academy of Sciences. He was an Associate Editor of Math Sys., Est. Contr. (91–93), Automatica (98–02), Asia J. Control. (2001–2003). Deputy Chief Editor of Control and Decision, J. Control Theory and App. etc. Since 2003 he has been Chairman of the Technical Committee on Control Theory, Chinese Automation Association. He is the author/co-author of over 150 journal papers and 5 books. His research interests include nonlinear systems and control, Numerical Method. He is elected IEEE Fellow in 2005.

Gang FENG received the B.E. and M.E. degrees in Automatic Control (of Electrical Engineering) from Nanjing Aeronautical Institute, China in 1982 and in 1984 respectively, and the Ph.D. degree in Electrical Engineering from the University of Melbourne, Australia in 1992. He has been with City University of Hong Kong since 2000 and was with School of Electrical Engineering, University of New South Wales, Australia, 1992–1999. He was awarded an Alexander von Humboldt Fellowship in 1997–1998. He was a visiting Fellow at National University of Singapore (1997), and Aachen Technology University, Germany (1997-1998). He has authored and/or coauthored more than 100 referred international journal papers and numerous conference papers. His current research interests include robust adaptive control, signal processing, piecewise linear systems, and intelligent systems and control. Dr. Feng is an associate editor of IEEE Trans. on Fuzzy Systems, and IEEE Trans. on Systems, Man & Cybernetics, Part C, Journal, of Control Theory and Applications, and was an associate editor of the Conference Editorial Board of IEEE Control System Society.



Zairong XI was born in Hunan, China, in 1969. He received the M.S. degree in mathematics from Zhengzhou University, Zhengzhou, China, in 1997, and the Ph.D. degree in control theory from the Chinese Academy of Sciences, Beijing, China, in 2000. He was a Postdoctoral Fellow in Tsinghua University, Beijing. In 2002, he joined the Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences. Now he is Associate Professor. His research interests include nonlinear systems analysis and synthesis, power systems control, mechanical systems, stability analysis, robust control, and quantum control.