[8] V. A. Ugrinovskii and I. R. Petersen, "Finite horizon minimax optimal control of stochastic partially observed time varying uncertain systems," *Math. Control, Signals, Syst.*, vol. 12, pp. 1–23, 1999.
[9] R. K. Boel, M. R. James, and I. R. Petersen, "Robustness and risk-sensitive filtering," *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 451–461, Mar. 2002.
[10] B. G. Leroux, "Maximum-likelihood estimation for hidden Markov models," *Stoch. Processes Appl.*, vol. 40, pp. 127–143, 1992.
[11] Y. E. Ephraim and N. Merhav, "Hidden Markov processes," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1518–1569, Jun. 2002.
[12] L. Xie, "Finite horizon robust state estimation of uncertain finite-alphabet hidden Markov models," Ph.D. dissertation, Univ. College, Univ. New South Wales, Canberra, Australia, July 2004.
[13] A. N. Shiryayev, *Probability*. New York: Springer-Verlag, 1984.
[14] P. Dupuis and E. Ellis, *A Week Convergence Approach to the Theory of Large Deviations*. New York: Wiley, 1997.
[15] M. D. Donsker and S. R. S. Varadhan, "Asymptotic evaluation of certain Markov process expectation for large time, IV," *Comm. Pur. Appl. Math.*, vol. 36, pp. 183–212, 1983.
[16] P. G. Hoel, S. C. Port, and C. J. Stone, *Introduction to Stochastic Processes*. Boston, MA: Houghton Mifflin, 1972.
[17] J. R. Norris, *Markov Chains*. Cambridge, U.K.: Cambridge Univ. Press, 1997.
[18] P. Billingsley, *Probability and Measure*, 3rd ed. New York: Wiley, 1995.
[19] L. D. Davisson, G. Longo, and A. Sgarro, "The error exponent for the noiseless encoding of finite ergodic: Markov sources," *IEEE Trans. Inf. Theory*, vol. IT-27, no. 4, pp. 431–438, Apr. 1981.
[20] Z. Rached, F. Alajaji, and L. Campbell, "The Kullback-Leibler divergence rate between Markov sources," *IEEE Trans. Inf. Theory*, vol. 50, no. 5, pp. 917–921, May 2004.
[21] T. S. Han and K. Kobayashi, *Mathematics of Information and Coding*. Providence, RI: Amer. Math. Soc., vol. 203.
[22] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.
[23] R. M. Gray, *Entropy and Information Theory*. New York: Springer-Verlag, 1990.
[24] M. N. Do, "Fast approximation of Kullback–Leibler distance for dependence trees and hidden Markov models," *IEEE Signal Process. Lett.*, vol. 10, no. 4, pp. 115–118, Apr. 2003.
[25] Y. S. Chow and H. Teicher, *Probability Theory, Independence, Interchangeability, Martingales*, 2nd ed. New York: Springer-Verlag, 1988.

Controllability of Switched Bilinear Systems

Daizhan Cheng

***Abstract*—The controllability of switched bilinear systems (SBLSs) is considered. Three kinds of controllabilities, including weak controllability, approximate controllability, and global controllability, are investigated one by one. Sets of easily verifiable sufficient conditions are obtained for each case, which are applicable to a large class of switched bilinear systems.**

***Index Terms*—Accessibility Lie algebra, Chow's theorem, controllability, switched bilinear system (SBLS).**

I. Introduction

In recent years, the switched systems have attracted considerable attention from the control community [12]. Most of the efforts have been focused on stability [1], [7], [3], stabilization [6], and controllability [8], [15], [17] of switched systems.

The controllability of nonlinear control systems has been discussed for long time, using Lie algebraic technique [2], [16]. The method has been used for switched linear systems [4]. The controllability of bilinear systems has also been investigated a lot [11], [13].

Using Lie algebraic technique, this note investigates the controllability of switched bilinear systems (SBLSs) of the form

$$\dot{x}=A_{\sigma(t)}x+\sum_{i=1}^{m}\Big(B^i_{\sigma(t)}x+c^i_{\sigma(t)}\Big)u_i:=A_{\sigma(t)}x+B_{\sigma(t)}ux+C_{\sigma(t)}u$$
$$x\in\mathbb{R}^n,\ u\in\mathbb{R}^m \quad (1)$$

where $\sigma(t):[0,\infty)\rightarrow\Lambda$ is a measurable right continuous mapping and $\Lambda=\{1,2,\ldots,N\}$. Controls $u(t)$ are piecewise constant functions.

Through this note, the state–space $\mathbb{R}^n$ can be replaced by any arcwise connected open sub-manifold, M, of $\mathbb{R}^n$. So in the sequel, any one of such M may be used without further explanation.

Note that in (1) and in the sequel we use brief notation $B_\lambda u$ for $\sum_{i=1}^m B^i_\lambda u_i$. In fact, this is the semi-tensor product $B_\lambda\ltimes u$ [5], where $B_\lambda=\begin{pmatrix}B^1_\lambda & \cdots & B^m_\lambda\end{pmatrix}$ is an $n\times mn$ matrix.

Three kinds of controllabilities are investigated in this note, namely, weak controllability, approximate controllability and global controllability. As one of the main tools, the accessibility Lie algebra of switched systems is defined and investigated. Roughly speaking, the main result in this note consists of: i) it is proved that as the accessibility Lie algebra has full rank the system is weakly controllable; ii) if in addition, certain symmetric condition is satisfied, the approximate controllability is obtained; and iii) if the system concerned is practically controllable, some additional local conditions assure the global controllability. The results ii) and iii) are applicable to a large class of switched bilinear systems, which satisfy the symmetric condition.

The note is organized as follows. Section II considers weak controllability. Section III studies approximate controllability. Global controllability is investigated in Section IV. Section V is the conclusion.

Manuscript received July 16, 2003; revised March 1, 2004 and October 21, 2004. Recommended by Associate Editor W. Kang. This work was supported part by the NNSF of China under Grants 60274010, 60343001, 60221301, and 60334040.

The author is with the Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, P. R. China (e-mail: dcheng@iss03.iss.ac.cn).

Digital Object Identifier 10.1109/TAC.2005.844897

II. Accessibility Lie Algebra and Weak Controllability

To begin with, we give some rigorous definitions for reachability and controllability.

Definition 2.1: Consider a SBLS.

i) For a given $x \in \mathbb{R}^n$, if there exist piecewise constant controls and a selected switching law $\sigma(t)$ such that the trajectory of the controlled switched system can be driven from x to y, then y is said to be in the reachable set of x. The reachable set of x is denoted by $R(x)$.

ii) y is said to be weakly reachable from x, if there exists a spline-trajectories of the system, which connect a finite set of points, $x := x_0, x_1, \ldots, x_s := y$ pairwise in either forward or backward ways. Precisely, either $x_{k-1} \in R(x_k)$ or $x_k \in R(x_{k-1})$, $k = 1, 2, \ldots, s$. The weak reachable set of x is denoted by $WR(x)$.

iii) A sub-manifold, $\mathcal{I}$, of M is called an invariant sub-manifold of a switched control system, if for any piecewise constant controls u_i, any switched law $\sigma(t)$, and any $x_0 \in \mathcal{I}$, the trajectory of the controlled switched system remains in $\mathcal{I}$, i.e., $x(x_0, u, \sigma(t), t) \subset \mathcal{I}, \forall t, \forall u$.

iv) An invariant sub-manifold $\mathcal{I}$ is called a controllable sub-manifold if for any two points $x, y \in \mathcal{I}$, $x \in R(y)$.

v) An invariant sub-manifold $\mathcal{I}$ is called a weakly controllable sub-manifold if for any two points $x, y \in \mathcal{I}$, $x \in WR(y)$.

The controllable sub-manifolds are closely related to the Lie algebra generated by the vector fields extracted from the systems. Similar to the classical (nonswitching) case, we define the accessibility Lie algebra for SBLS as

Definition 2.2: For system (1), the accessibility Lie algebra is defined as

$$\mathcal{L}_a := \left\{ A_\lambda x, B_\lambda^i x + c_\lambda^i, \middle| \lambda \in \Lambda, i = 1, \ldots, m \right\}_{LA}. \tag{2}$$

The following result about weak controllability is a mimic of the corresponding result about general control systems [13], [16].

Proposition 2.3: The system (1) is globally weakly controllable, if the accessibility Lie algebra has full rank. That is

$$\operatorname{rank}(\mathcal{L}_a(x)) = n \qquad \forall x \in \mathbb{R}^n. \tag{3}$$

If (3) is satisfied, as for nonswitching case, it is said that the accessibility rank condition is satisfied.

Proof: According to Chow's theorem [2], for each x the weakly reachable set of x is the largest integral manifold of $\mathcal{L}_a$, passing through x. By condition (3), it contains an open neighborhood U of x. So, for any x_0 its weakly reachable set is an open set. We claim that $WR(x_0)$ is also closed. Otherwise, there exists a $x_1 \in \mathrm{cl}(WR(x_0)) \backslash WR(x_0)$. However, $WR(x_1)$ contains an open neighborhood, U, of x_1, so $U \cap WR(x_0) \neq \phi$. Say, $\xi \in U \cap WR(x_0)$. Then, $x \in WR(x_1)$, by symmetry, $x_1 \in WR(\xi)$, and $\xi \in WR(x_0)$. By transitivity, $x \in WR(x_0)$. Therefore, $U \subset WR(x_0)$, which is a contradiction. Now since $\mathbb{R}^n$ is arc-wise connected, its only nonempty closed-open set is M itself. That is, $WR(x_0) = \mathbb{R}^n$. □

Remark 2.4:

1) Weak controllability is based on an equivalent relation: weak reachability (WR) [13]. That is, i) $x \in WR(x)$; ii) if $x \in WR(y)$, then $y \in WR(x)$; iii) if $x \in WR(y)$ and $y \in WR(z)$, then $x \in WR(z)$. (This fact has been used in the previous proof.) So, the whole state–space is partitioned into weakly controllable subsets.

2) Since each switched model of an SBLS is analytic, using (generalized) Frobenius' theorem [14] and Chow's theorem, one sees easily that for system (1) the state–space M is partitioned into maximal connected weak controllability sub-manifolds. Each weak controllability sub-manifold is the maximal integral manifold of $\mathcal{L}_a$. This is a very important topological structure for weakly controllable sub-manifolds of SBLSs.

3) Definitions 2.1 and 2.2 and Proposition 2.3 can be extended to general switched control systems. Some of the following obtained topological structures of the weakly controllable sub-manifolds of SBLS are also applicable to analytic control systems.

Definition 2.5: Let V be a set of vector fields. V is said to be k-symmetric, if for any vector field $X \in V$, there is a vector field $Y \in V$ with $Y = -kX$, $k > 0$.

Remark 2.6: Using Chow's theorem one sees that for SBLSs if $\mathcal{L}_a$ has a k-symmetric generator of the form $\{f_\lambda + g_\lambda u_\lambda\}$, then its weakly controllable sub-manifolds become controllable sub-manifolds.

The following lemma is useful in the sequel.

Lemma 2.7: Let Δ be an involutive analytic distribution, i.e., an involutive distribution generated by analytic vector fields. Moreover, assume $x \in WR(y)$ via spline integral curves of Δ, then

$$\operatorname{rank}(\Delta(y)) = \operatorname{rank}(\Delta(x)).$$

Proof: Without loss of generality, we assume there exists a flow $\phi(t)$ of $Y \in \Delta$ such that $x = \phi(0)$ and $y = \phi(t)$ and let $X \in \Delta$. Then, using the Campbell–Baker–Hausdorff formula [14]

$$\phi(t)_*(X(x)) = \sum_{i=0}^{\infty} \frac{1}{i!} t^i ad_Y^i X(y).$$

Since X is arbitrary and $\phi(t)_*$ is an isomorphism, one sees that $\operatorname{rank}(\Delta_x) \leq \operatorname{rank}(\Delta_y)$. Using the formula for negative time, we have the reversed inequality. □

Then, we have the following result.

Proposition 2.8:

1) Consider a SBLS. For a given point $x \in \mathbb{R}^n$ if $\operatorname{rank}(\mathcal{L}(x_0)) = m$ and there exists an open neighborhood U of x_0 such that

$$S(x) := \{x \in U \mid \operatorname{rank}(\mathcal{L}_a(x)) = m\}$$

is an mth dimensional regular sub-manifold of U, then $S(x)$ is a weak controllable sub-manifold.

2) If $\mathcal{L}_a$ has a k-symmetric generator of the form $\{f_\lambda + g_\lambda u_\lambda\}$, $S(x)$ is a controllable sub-manifold.

Proof:

1) Note that by the Frobenius' theorem [refer to 2) of Remark 2.4], for a bilinear system the m-th degree largest integral manifold passing through x_0 always exists. By Lemma 2.7, for any point y on this manifold, $\mathcal{L}_a(y) = m$. Now since $S(x)$ is the unique mth degree regular sub-manifold, it must be contained in the largest integral manifold.

2) Using Remark 2.6 and the same argument in 1), 2) is obvious. □

Remark 2.9:

1) The advantage of Proposition 2.8 lies on that, instead of searching for the controllable sub-manifold we have only to check the rank condition. The problem is then tremendously simplified.

Example 2.10: Consider the following switched system:

$$\dot{x} = B_{\sigma(t)} u x, \qquad x \in \mathbb{R}^2 \backslash \{0\} \tag{4}$$

where $\Lambda = \{1, 2\}$ and

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

It is easy to see that

$$\{B_1, B_2\}_{LA} = \mathrm{Span}\{B_1, B_2\}.$$

So, we have only to consider the rank of the distribution generated by these two vector fields. Observe that

$$\det\begin{pmatrix} B_1 x & B_2 x \end{pmatrix} = (x_1 + x_2)(x_1 - 2x_2).$$

According to Remark 2.6 and Proposition 2.8, one sees that the controllable sub-manifolds consist of eight components: four angular regions, which are two dimensional sub-manifolds, and four rays, which are one dimensional sub-manifolds. They form a partition of $\mathbb{R}^2\backslash\{0\}$, which are split by two lines: $x_1 = -x_2$ and $x_1 = 2x_2$. Namely, four two-dimensional controllable sub-manifolds are

$$\left\{x \in \mathbb{R}^2\backslash\{0\} \mid x_1 > 0, -x_1 < x_2 < \frac{1}{2}x_1\right\}$$
$$\left\{x \in \mathbb{R}^2\backslash\{0\} \mid x_1 < 0, \frac{1}{2}x_1 < x_2 < -x_1\right\}$$
$$\{x \in \mathbb{R}^2\backslash\{0\} \mid x_2 > 0, -x_2 < x_1 < 2x_2\}$$
$$\{x \in \mathbb{R}^2\backslash\{0\} \mid x_2 < 0, 2x_2 < x_1 < -x_2\}$$

and four one-dimensional controllable sub-manifolds are

$$\left\{x \in \mathbb{R}^2\backslash\{0\} \mid x_1 > 0, x_2 = \frac{1}{2}x_1\right\}$$
$$\left\{x \in \mathbb{R}^2\backslash\{0\} \mid x_1 < 0, x_2 = \frac{1}{2}x_1\right\}$$
$$\{x \in \mathbb{R}^2\backslash\{0\} \mid x_1 > 0, x_2 = -x_1\}$$
$$\{x \in \mathbb{R}^2\backslash\{0\} \mid x_1 < 0, x_2 = -x_1\}.$$

□

III. Approximate Controllability

This section considers the approximate controllability of (1). Approximate controllability means the state can be driven to approach a given destination point. In some journals it is also called eventual controllability. It is practically useful. We give a rigorous definition.

Definition 3.1: System (1) is said to be approximately controllable at $x \in M$ if for any $y \in M$ and any given $\epsilon > 0$, there exist suitable controls and switching law such that the spline-trajectories of the switched controlled models can reach the ϵ neighborhood of y. The system is said to be approximately controllable if it is approximately controllable at every $x \in M$.

The constructive nonlinear decomposition technique has been used widely for bilinear systems [10], [11]. For this approach, instead of studying the switched bilinear system (1), we consider the following two switched systems: A linear system without control and two switched bilinear homogeneous control systems as follows:

$$\dot{x} = A_{\sigma(t)}x \tag{5}$$
$$\dot{x} = B_{\sigma(t)}ux + C_{\sigma(t)}u \quad \dot{x} = -\left(B_{\sigma(t)}ux + C_{\sigma(t)}u\right). \tag{6}$$

Denote by $R_{LH}(x_0)$ the reachable set of the spline trajectories of (5) and (6). Then, we have the following result, which is due to [11] for BLS. It can be extended to SBLS without any difficulties.

Lemma 3.2: [11] Consider system (1). For every $x_0 \in \mathbb{R}^n$, denote the reachable set of x_0 by $R(x_0)$, then

$$R_{LH}(x_0) \subset \mathrm{cl}\{R(x_0)\}. \tag{7}$$

Here, cl is used to denote the closure of a set. Denote

$$\mathcal{V}_c = \left\{(B_\lambda^i x + c_\lambda^i)\Big| \lambda \in \Lambda, i = 1, \ldots, m\right\}_{LA}$$

which is generated by the vectors of input channels.

Using Lemma 3.2, we have the following result immediately.

Proposition 3.3: Consider system (1). If

$$\mathrm{rank}(\mathcal{V}_c(x)) = n \qquad \forall x \in \mathbb{R}^n \tag{8}$$

then the system is approximately controllable.

Proof: We claim that the complement of reachable set, denoted by $R^c(x_0)$, is nowhere dense. Otherwise, there exists a nonempty open set $O \neq \phi$, $O \subset R^c(x_0)$. Then, $O^c \supset R(x_0)$. Since O^c is a closed set, we have $O^c \supset cl\{R(x_0)\}$. It follows from (7) that

$$O \cap R_{LH}(x_0) = \phi.$$

On the other hand, the generator of $\mathcal{V}_c$ is from (6), which is a symmetric set of vector fields. According to Proposition 2.3 and the symmetry, (7) implies that $R_{LH}(x_0) = \mathbb{R}^n$. This fact leads to a contradiction.

Now, given $y \in \mathbb{R}^n$, for any $\epsilon > 0$, denote its ϵ neighborhood by $B_\epsilon(y)$. Since $R^c(x_0)$ is nowhere dense, there exits $y_0 \in B_\epsilon(y)$ which is also in $R(x_0)$, which completes the proof. □

To avoid the obstacle of nonsymmetry of drift terms, we consider a class of systems, which, with properly chosen state feedback on every switching model, have k-symmetric drift terms.

Definition 3.4: The system (1) is said to have a feedback k-symmetric drift terms, if there exist controls u_λ^0, $\lambda \in \Lambda$, such that the new drift terms under feedback

$$\tilde{A}_\lambda = A_\lambda + B_\lambda u_\lambda^0, \qquad \lambda \in \Lambda \tag{9}$$

form a k-symmetric set

$$\tilde{\mathcal{A}} := \left\{\tilde{A}_\lambda | \lambda \in \Lambda\right\}.$$

Remark 3.5: Consider an affine switched nonlinear system

$$\dot{x} = f_{\sigma(t)}(x) + \sum_{i=1}^{m} g_{\sigma(t)}^i U_i, \qquad \sigma(t) \in \{1, 2, \ldots, N\}. \tag{10}$$

Equation (10) has feedback k-symmetric drift terms iff for each i there exist a j and a positive real number $k > 0$, such that

$$f_i + kf_j \in \mathrm{Span}\{g_s^t\big| s = i, j; t = 1, \ldots, m\}.$$

A sufficient condition for feedback k-symmetry is

$$\dim\left(\mathrm{Span}\{g_s^t\big| s = i, j; t = 1, \ldots, m\}\right) = n. \tag{11}$$

It is easy to see that a lot of practical systems satisfy (11).

Similar to Proposition 3.3, we can prove the following.

Proposition 3.6: Consider system (1). Assume

i) the system has feedback k-symmetric drift terms;

ii)

$$\dim\{\mathcal{L}_a(x)\} = n \qquad \forall x \in \mathbb{R}^n.$$

Then, the system is practically controllable.

Proof: Using a prefeedback, we can replace (5) by

$$\dot{x} = \tilde{A}_{\sigma(t)}x. \tag{12}$$

Then, the conditions i) and ii) imply that for any $x_0 \in \mathbb{R}^n$ the set $R_{LH}(x_0)$ corresponding to (12)–(6) is $\mathbb{R}^n$. The rest argument is the same of the one for Proposition 3.3. □

Remark 3.7:

1) If the feedback k-symmetric drift terms form a generator of $\mathcal{L}_a$, according to Remark 2.6, the system is globally controllable.
2) In fact, if there is a subset $\Lambda' \subset \Lambda$, such that the corresponding feedback vector fields

$$\tilde{\mathcal{A}}' := \left\{ \tilde{A}_{\lambda'} | \lambda' \in \Lambda' \right\}$$

form a symmetric set, then we define

$$\mathcal{L}' := \left\{ A_{\lambda'} x, B_\lambda^j x + c_\lambda^j \middle| j = 1, \ldots, m; \lambda' \in \Lambda', \lambda \in \Lambda \right\}_{LA}.$$

It is clear that the switched bilinear system (1) is globally approximately controllable if

$$\dim\{\mathcal{L}'(x)\} = n \qquad \forall x \in \mathbb{R}^n.$$

Example 3.8: Consider a bilinear switched system

$$\dot{x} = A_{\sigma(t)} x + u \left(B_{\sigma(t)} x + C_{\sigma(t)} \right) \tag{13}$$

where $\Lambda = \{1, 2\}$ and

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad C_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A straightforward computation shows that rank $(\mathcal{L}_a(x)) = 3$, $\forall x \in \mathbb{R}^3$. i.e., the accessibility rank condition is satisfied.

In addition, it is obvious that (13) has feedback k- symmetric drift terms. According to Proposition 3.6, the system is approximately controllable. □

Remark 3.9: It is easy to see that the main result obtained in this section can be extended to affine nonlinear systems.

IV. Global Controllability

This section considers the global controllability. Recall a result of local controllability for general control systems first.

Consider a general control system

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n \tag{14}$$

where f is a C^1 mapping. Let x_0 be an equilibrium of the control system with control $u_e(x)$, i.e., $f(x_e, u_e(x_e)) = 0$. Define

$$E = \frac{\partial f}{\partial x}(x, u_e(x))\bigg|_{x_0, u_e(x_0)}, \quad D = \frac{\partial f}{\partial u}(x, u_e(x))\bigg|_{x_0, u_e(x_0)}. \tag{15}$$

We have the following sufficient condition for local controllability.

Lemma 4.1: [13] Consider system (14). Assume there exist $x_e \in \mathbb{R}^n$ and control $u_e(x)$, such that, $f(x_e, u_e(x_e)) = 0$. Moreover, assume (E, D), defined in (15), is completely controllable. Then, (14) is locally controllable at x_e. That is, there exists an open neighborhood U of x_e, such that for any $x, y \in U$, $x \in R(y)$ and $y \in R(x)$.

Using it and the approximate controllability investigated in last section, we deduce some sufficient conditions for global controllability.

Definition 4.2: Consider a bilinear system

$$\dot{x} = Ax + Bux + Cu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m. \tag{16}$$

1) A pair $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium pair, if

$$Ax_e + Bu_e x_e + Cu_e = 0. \tag{17}$$

2) An equilibrium pair (x_e, u_e) is said to be stable if $A + Bu_e$ is Hurwitz, it is said to be antistable if $-(A + Bu_e)$ is Hurwitz.
3) An equilibrium pair (x_e, u_e) is said to be controllable, if $(A + Bu_e, B(I_m \otimes x_e) + C)$ is a controllable pair.

Theorem 4.3: Consider system (1). Assume

i) it is approximately controllable;
ii) there exist $\lambda_1 \in \Lambda$ and an equilibrium pair $(x_e, u_{\lambda_1}^e)$, such that $(x_e, u_{\lambda_1}^e)$ is antistable for the λ_1th switching model;
iii) there exist $\lambda_2 \in \Lambda$ ($\lambda_2 = \lambda_1$ is allowed) and an equilibrium pairs $(x_e, u_{\lambda_2}^e)$, such that $(x_e, u_{\lambda_2}^e)$ is controllable for the λ_2th switching model.

Then, (1) is globally controllable.

Proof: Since $(x_e, u_{\lambda_2}^e)$ is controllable, by Lemma 4.1 there exists a neighborhood U of x_e such that the λ_2th switching model is controllable over U.

Next, we show that for any $x, y \in \mathbb{R}^n$ we can drive the state from x to y. Since the system is approximately controllable we can first drive x to a point $\xi \in U$. Denote the vector field of the closed-loop λ_1th switching model with control $u_{\lambda_1}^e$ by V, that is

$$V = (A_{\lambda_1} + B_{\lambda_1} u_{\lambda_1}^e) x + C_{\lambda_1} u_{\lambda_1}^e = (A_{\lambda_1} + B_{\lambda_1} u_{\lambda_1}^e)(x - x_e).$$

Since $-V$ is stable, so the integral curve of $-V$ goes from y to x_e asymptotically. Hence, there is a $T > 0$ such that

$$e_T^{-V}(y) = \eta \in U.$$

Equivalently

$$y = e_T^V(\eta).$$

To complete the proof, we have only to drive the state from ξ to η. This can be done by choosing λ_2th switching model and a suitable control u^{λ_2}, because of local controllability of this model over U.

Summarizing this argument, we can drive x to y in three steps.

Step 1) According to approximate controllability, we can drive x to $\xi \in U$.
Step 2) According to the local controllability, we can drive ξ to η.
Step 3) According to the antistability, we can drive η to y by feedback system (vector field) V.

□

Example 4.4: Recall Example 3.8. We prove that (13) is globally controllable. Using Theorem 4.3, we have to check conditions i)–iii). i) is proved in Example 3.8. Now, we choose a pair as $(x_e, u_e) = (0, 0)$. Obviously, it is an equilibrium pair. We then show that for the first model it is antistable. In fact

$$A_1 + B_1 u_e = A_1$$

which is antistable. So, ii) is satisfied. Still use this pair to the first model. We have

$$(A_1 + B_1 u_e, B_1 x_e + C) = (A_1, c_1). \tag{18}$$

It is easy to check that (18) is completely controllable, which implies iii). The conclusion follows. □

Following the same train of thought as in the proof of Theorem 4.3, we can have the following result immediately.

Proposition 4.5: Consider system (1). Assume that

i) there exist $\lambda_1 \in \Lambda$ and an equilibrium pair $(x_e, u_{\lambda_1}^e)$, such that $(x_e, u_{\lambda_1}^e)$ is stable for the λ_1th switching model;

ii) there exist $\lambda_2 \in \Lambda$ and an equilibrium pair $(x_e, u^e_{\lambda_2})$, such that $(x_e, u^e_{\lambda_2})$ is antistable for the λ_2th switching model;

iii) there exist $\lambda_3 \in \Lambda$ and an equilibrium pairs $(x_e, u^e_{\lambda_3})$, such that $(x_e, u^e_{\lambda_3})$ is controllable for the λ_3th switching model.

Then, (1) is globally controllable.

Remark 4.6: In fact, in Theorem 4.3 x_e in condition ii) [specified as x_e^2 to distinguish it from x_e in condition iii)] can be different from the x_e (x_e^3) in condition iii). It is enough that $x_e^2 \in R(x_e^3)$. Particularly, since $R(x_e^3)$ contains a controllable open neighborhood, U, of x_e^3, so it suffices that $x_e^2 \in U$. Similarly, for Proposition 4.5, $x_e^3 \in R(x_e^1)$ and $x_e^2 \in R(x_e^3)$ are enough for the global controllability. Particularly, when U is a controllable open neighborhood of x_e^3, then $x_e^1, x_e^2 \in U$ is enough for the global controllability.

V. Conclusion

In this note, the controllability of a switched bilinear control system was considered. Three kinds of controllabilities, namely, weak controllability, approximate controllability, and global controllability, are considered. It was proved that accessibility rank condition assures weak controllability, weak controllability plus feedback k-symmetry implies approximate controllability and approximate controllability plus local controllability and the existence of stable and antistable equilibriums imply global controllability. Since it was shown that a large class of systems satisfy the condition of feedback k-symmetry, the results cover a large class of switched bilinear systems.

References

[1] A. A. Agrachev and D. Liberzon, "Lie-algebraic stability criteria for switched systems," *SIAM J. Control Opt.*, vol. 41, no. 1, pp. 253–269, 2001.

[2] R. W. Brockett, "Nonlinear systems and differential geometry," *Proc. IEEE*, vol. 64, no. 1, pp. 61–71, Jan. 1976.

[3] D. Cheng, L. Guo, and J. Huan, "On quadratic Lyapunov functions," *IEEE Trans. Autom. Control*, vol. 48, no. 5, pp. 885–890, May 2003.

[4] D. Cheng, "Accessibility of switched linear systems," in *Proc. 42nd IEEE Conf. Decision Control*, Mauii, HI, 2003, pp. 5759–5764.

[5] D. Cheng, X. Hu, and Y. Wang, "Non-regular feedback linearization of nonlinear systems via a normal form algorithm," *Automatica*, vol. 40, no. 3, pp. 439–447, 2004.

[6] D. Cheng, "Stabilization of planar switching systems," *Syst. Control Lett.*, vol. 51, no. 2, pp. 79–88, 2004.

[7] W. P. Dayawansa and C. F. Martin, "A converse Lyapunov theorem for a class of dynamical systems which undergo switching," *IEEE Trans. Autom. Control*, vol. 44, no. 4, pp. 751–760, Apr. 1999.

[8] J. Ezzine and A. H. Haddad, "Controllability and observability of hybrid systems," *Int. J. Control*, vol. 49, no. 6, pp. 2045–2055, 1989.

[9] A. Isidori, *Non-Linear Control Systems*, 3nd ed. New York: Springer-Verlag, 1995.

[10] V. Jurdjevic and G. Sallet, "Controllability properties of affine systems," *SIAM J. Control Opt.*, vol. 22, no. 3, pp. 501–508, 1984.

[11] A. Y. Khapalov and R. R. Mohler, "Reachable sets and controllability of bilinear time-invariant systems: A qualitative appraoch," *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1342–1346, Sep. 1996.

[12] D. Liberzon and A. S. Morse, "Basic problems on stability and design of switched systems," *IEEE Control Syst. Mag.*, vol. 19, no. 5, pp. 59–70, Oct. 1999.

[13] R. R. Mohler, *Nonlinear Systems, Volume II, Applications to Bilinear Control*. Upper Saddle River, NJ: Prentice-Hall, 1991.

[14] P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed. New York: Springer-Verlag, 1993.

[15] Z. Sun, S. S. Ge, and T. H. Lee, "Controllability and reachability criteria for switched linear systems," *Automatica*, vol. 38, pp. 775–786, 2002.

[16] H. J. Susmann, "A general theorem on local controllability," *SIAM J. Control Opt.*, vol. 25, pp. 158–194, 1987.

[17] G. Xie and L. Wang, "Controllability and stabilizability of switched linear systems," *Syst. Control Lett.*, vol. 42, no. 2, pp. 135–155, 2003.

Robust Normalization and Stabilization of Uncertain Descriptor Systems With Norm-Bounded Perturbations

Chong Lin, Qing-Guo Wang, and Tong Heng Lee

***Abstract*—This note is concerned with the problem of the so-called quadratic normalization and stabilization via both proportional and derivative state feedback (PDSF) and proportional and derivative output feedback (PDOF) for uncertain descriptor systems with norm-bounded perturbations. Necessary and sufficient conditions are presented in terms of linear matrix inequalities (LMIs) or quadratic matrix inequalities (QMIs), by using a simple idea of changing the problem to the corresponding stabilization problem of an augmented uncertain system.**

***Index Terms*—Descriptor systems, norm-bounded perturbation, normalization, stabilization.**

I. Introduction

Consider the following linear descriptor system:

$$E\dot{x}(t) = Ax(t) + Bu(t) \quad y(t) = Cx(t) \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$ with rank $(E) \leq n$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$. Such a system arises in a variety of physical systems such as electrical circuits, moving robots and many other systems which can be modeled by dynamic equations and algebraic constraints. It is well known that the regularity of system (1) (or, the pair (E, A)), i.e., $\det(sE-A) \neq 0$, guarantees the existence and uniqueness of solutions to (1) on $[0, \infty)$, and the condition $\deg(\det(sE - A)) =$ rank E ensures that system (1) is impulse-free, i.e., there is no impulse behavior in the system [4]. Note that the system may have initial jump for noncompatible initial conditions [11].

It is known that, if the derivative of the state $x(t)$ [and, thus, of the output $y(t)$] is available, a proportional and derivative feedback may render the closed-loop system of (1) a normal system (i.e., the normalization problem). The use of a proportional and derivative feedback has a well engineering motivation and so far there have been many research papers published to address the importance and the engineering motivation of proportional and derivative state feedback (PDSF) and proportional and derivative output feedback (PDOF) (see [1], [3], [5], [6], [8], [13], and the references therein). A PDSF could make the closed-loop system regular and impulse-free (i.e., the regularization problem) [1], [13]. Such a regularization problem also has a complete solution by using PDOF [3]. These regularization results are based on Kronecker canonical decomposition or orthogonal matrix transformation, and so are hard to apply for further simultaneously solving the stability problem. While PDSF could regularize a descriptor system, it would also be useful for the pole assignment. Constructions of PDSF matrices are given in [8] and [5] to shift all open-loop poles to desired finite points under certain conditions. All of the aforementioned developments using proportional and derivative feedback need matrix decompositions, and these methods could not be applicable anymore if the system matrices subject to perturbations. Recently, PDSF H_∞ control approach is proposed in [6] for descriptor systems. Still, the method is based on matrix decompositions, and is not applicable for perturbed systems.

Manuscript received October 1, 2003; revised May 23, 2004 and September 29, 2004. Recommended by Associate Editor M. Kothare.

The authors are with the Department of Electrical and Computer Engineering, National University of Singapore, Singapore 119260, Singapore (e-mail: elewqg@nus.edu.sg).

Digital Object Identifier 10.1109/TAC.2005.844908