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Optimization leads to symmetry

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Abstract: The science of complexity studies the behavior and properties of complex systems in nature and human society. Particular interest has been put on their certain simple common properties. Symmetry is one of such properties. Symmetric phenomena can be found in many complex systems. The purpose of this paper is to reveal the internal reason of the symmetry. Using some physical systems and geometric objects, the paper shows that many symmetries are caused by optimization under certain criteria. It has also been revealed that an evolutional process may lead to symmetry.

Keywords: Complex system; Symmetry; Optimization; Evolutional process

1 Symmetry - a property of complex systems

An important objective of the science of complexity is to reveal the properties of complex systems and to find its causes so as to apply it. Self-organization, power law etc are properties of complex systems which are founded in the real world [1]. Small world, scale free, etc. are characteristics of networks [2]. Symmetry, which is familiar to most people, is also an important nature of the complex world and has been studied by many researchers. What is symmetry? [3] gives an informal definition as follows: "The idea of symmetry can be viewed in very different ways: The narrowest interpretation is limited to two-sided symmetry, as applied more or less exactly, to the external form of the human body. The broadest interpretation understands by symmetry the property of anything that is in some way regular and shows repetitions."

In modern mathematics symmetry is explained as a kind of invariance. Precisely, a set of objects is said to hold a symmetry if there exists a group, such that the geometric shapes or certain properties of the set are invariant under group action.

Symmetry is also important for dynamic systems and particularly for control systems [4]. A natural problem is: How the symmetry of complex systems is caused? There are few researches which have been done for this. A commonly observed phenomenon is: most creatures are highly symmetric. Is it occasional? The answer is "No". In the process of evolution, there are some un-symmetric creatures. But in the process, the strong competitive power to survive competition has enabled the creatures with symmetric bodies to succeed. In this sense, the optimization of nature selected symmetry.

Among the planar graphs with a fixed perimeter, circle encloses the largest area. Among the bodies with a fixed surface area, sphere encloses the largest volume. These are well known facts and prove that optimization leads to symmetry.

The purpose of this paper is to explore and further prove this assertion: optimization leads to symmetry. The rest of this paper is organized as follows: Section 2 investigates some physical systems to see that minimum energy leads to symmetry; Geometric symmetry is studied in Section 3; In Section 4, we use an example to show that certain symmetries are the results of dynamic evolutions.

2 Symmetries of physical systems

In this section, we study some simple examples of physical systems to prove that minimizing energy leads to symmetry.

Example 2.1 Suppose there are n resistances $R_1, R_2, \cdots, R_n$ that are connected in the parallel way and satisfy the condition that $\sum_{i=1}^{n} R_i = R$ (Fig. 1). If the voltage U pressing upon them is constant, then the total power P dissipated is $P = U^2 \sum_{i=1}^{n} \frac{1}{R_i}$. Now consider such a question: what conditions R_i should be satisfied to minimize the total power P? In order to solve this problem, we first define a Lagrange function

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$$L = U^2 \sum_{i=1}^{n} \frac{1}{R_i} + \lambda\Big(\sum_{i=1}^{n} R_i - R\Big),$$

where λ is the Lagrange constant. Set $\frac{\partial L}{\partial R_i} = 0, i = 1,2,\cdots,n$, we have

$$-U^2 \frac{1}{R_i^2} + \lambda = 0 \quad (i = 1,2,\cdots,n).$$

So $R_1 = R_2 = \cdots = R_n = \frac{R}{n}$. It is easy to verify that under these conditions the total power P reaches the minimum. So we can say that the distribution of R_i dissipating the least energy is symmetric.

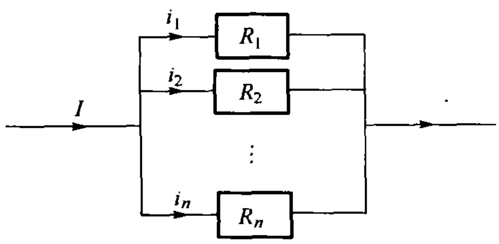

Fig. 1 Symmetry of circuit.

Example 2.2 Consider the conical pendulum (see Fig.2). The pendulum string, whose length is l, can slide freely. $l_1 + l_2 = l$. The pendulum masses are m_1 and m_2 respectively and satisfy $m_1 + m_2 = M$. The axis is rotating with an angular velocity ω. It is easy to verify that the whole kinetic energy of the pendulum bobs is

$$E = \frac{1}{2} m_1 \omega^2 l_1^2 \sin^2\theta_1 + \frac{1}{2} m_2 \omega^2 l_2^2 \sin^2\theta_2,$$

where

$$\begin{cases} T = m_1\omega^2 l_1 = m_2 \omega^2 l_2, \\ T\cos\theta_1 = m_1 g, \\ T\cos\theta_2 = m_2 g, \end{cases}$$

and T is the tensile force. The above equations together with the conditions about pendulum length and mass induce the following constraints:

$$\begin{cases} m_1 l_1 - m_2 l_2 = 0, \\ m_1\cos\theta_2 - m_2\cos\theta_1 = 0, \\ \omega^2 l_2 \cos\theta_2 - g = 0, \\ l_1 + l_2 - l = 0, \\ m_1 + m_2 - M = 0. \end{cases} \tag{1}$$

Define a Lagrange function as:

$$\begin{aligned} L = &\frac{1}{2} m_1\omega^2 l_1^2 \sin^2\theta_1 + \frac{1}{2} m_2 \omega^2 l_2^2 \sin^2\theta_2 \\ &+ \lambda(m_1 l_1 - m_2 l_2) + \mu(m_1\cos\theta_2 - m_2\cos\theta_1) \\ &+ \rho(\omega^2 l_2\cos\theta_2 - g) + \xi(l_1 + l_2 - l) \\ &+ \beta(m_1 + m_2 - M). \end{aligned}$$

Let

$$\begin{cases} \frac{\partial L}{\partial m_1} = \frac{1}{2}\omega^2 l_1^2 \sin^2\theta_1 + \lambda l_1 + \mu\cos\theta_2 + \beta = 0, \\ \frac{\partial L}{\partial m_2} = \frac{1}{2}\omega^2 l_2^2 \sin^2\theta_2 - \lambda l_2 - \mu\cos\theta_1 + \beta = 0, \\ \frac{\partial L}{\partial l_1} = m_1\omega^2 l_1 \sin^2\theta_1 + \lambda m_1 + \xi = 0, \\ \frac{\partial L}{\partial l_2} = m_2\omega^2 l_2 \sin^2\theta_2 - \lambda m_2 + \rho\omega^2\cos\theta_2 + \xi = 0, \\ \frac{\partial L}{\partial \cos\theta_1} = -m_1\omega^2 l_1^2\cos\theta_1 - \mu m_2 = 0, \\ \frac{\partial L}{\partial \cos\theta_2} = -m_2\omega^2 l_2^2 \cos\theta_2 + \mu m_1 + \rho\omega^2 l_2 = 0. \end{cases} \tag{2}$$

Combining (1) and (2), we have

$$\begin{cases} m_1 = m_2 = \frac{M}{2},\ l_1 = l_2 = \frac{l}{2}, \\ \cos\theta_1 = \cos\theta_2 = \frac{2g}{\omega^2 l},\ \lambda = \frac{2g^2}{\omega^2 l}, \\ \mu = -\frac{gl}{2},\ \rho = \frac{Mg}{\omega^2}, \\ \xi = -\frac{M\omega^2 l}{4},\ \beta = -\frac{\omega^4 l^2 - 4g^2}{8\omega^2}. \end{cases} \tag{3}$$

It is easy to see that under constraint (1), the total kinetic energy reaches the minimum when $m_1 = m_2, l_1 = l_2, \theta_1 = \theta_2$. This example also shows that the least kinetic energy leads to symmetry.

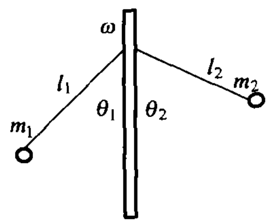

Fig. 2 Symmetry of flexible conical pendulum.

3 Symmetries in geometry

In this section we consider some geometric symmetries deduced by optimizing certain quantities.

Example 3.1 Consider inscribed polygons of n sides of a circle (Fig.3). It can be proved that the inscribed n polygon of the largest area must be regular. To interpret this, one only needs to prove that if the sum $\theta_1 + \theta_2 = \theta_0$ is a constant, the sum area of the two triangles ΔOAB and ΔOBC reaches maximum if $\theta_1 = \theta_2$. In fact, the whole area is

$$S = \frac{1}{2}R^2\sin\theta_1 + \frac{1}{2}R^2\sin\theta_2$$

$$= R^2 \sin\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}$$
$$= R^2 \sin\theta_0 \cos\frac{\theta_1-\theta_2}{2}.$$

Obviously when $\theta_1 = \theta_2$, the area reaches the maximum.

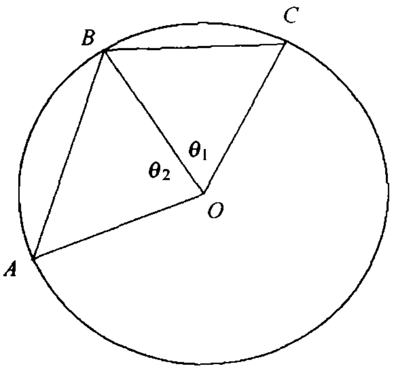

Fig. 3 Maximum area of two sector triangles.

Example 3.2 Consider inscribed tetrahedrons of a sphere (Fig.4). Firstly, if the points A, B, C are fixed, it is easy to check that the volume of tetrahedron D-ABC is the largest if D is on the line which is perpendicular to ΔABC and through the circumcenter of it, and here the height $h = OO' + R$. In fact, if there is another point D' on the sphere other than D, then the height of tetrahedron D'-ABC is $h' = OO' + R\sin\theta < h$, so $V_{D'-ABC} < V_{D-ABC}$; Secondly, if we fix the acme D of tetrahedron D-ABC and let points A, B, C moving on the plane determined by the original points A, B, C, then one sees easily that when ΔABC is the inscribed equilateral triangle of circle O', $S_{\Delta ABC}$ is of the largest area (see Example 3.1) and so is the volume $V_{D\text{-}ABC}$.

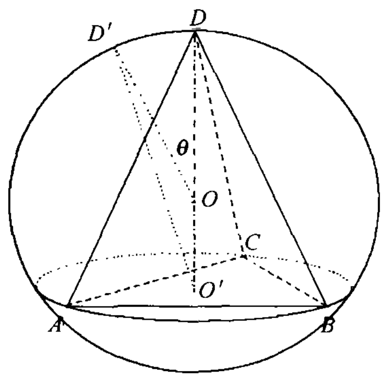

Fig. 4 Maximum volume of circumscribed pyramid.

Similarly, each face should be an equilateral triangle. Therefore, the tetrahedron $ABCD$ with largest volume is a regular one.

4 Symmetry as a result of evolution

It was mentioned that the evolution of creatures in the nature leads to symmetry of bodies. In this section we use an example to describe a symmetry generated by a dynamic evolution.

Cyclic pursuit is a long discussed problem [5 ~ 7]. We modify the simple pursuit to an attraction-repulsion process, which is similar to [8].

Example 4.1 Assume there are three particles on a plane, say P_1, P_2, P_3, P_i pursuits P_{i+1} when $\| P_i - P_{i+1} \| > \delta$; otherwise, P_i repulses from P_{i+1} (For notational ease, $P_4 := P_1$). The model is described as

$$\dot{P}_i = (P_{i+1} - P_i)(\| P_{i+1} - P_i \| - \delta),\ i = 1,2,3. \tag{4}$$

We want to show that as $t \to \infty$, the triangle $\Delta P_1 P_2 P_3$ will converge to an equilateral triangle.

Denote by $z_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} = P_{i+1} - P_i (i = 1,2,3)$, then we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}(\sqrt{x_{i+1}^2 + y_{i+1}^2} - \delta) - \begin{pmatrix} x_i \\ y_i \end{pmatrix}(\sqrt{x_i^2 + y_i^2} - \delta),\ i = 1,2,3.$$

Define

$$r_i = \sqrt{x_i^2 + y_i^2} - \delta,\ i = 1,2,3.$$

If we can prove that $r_i(t) \to 0$ as $t \to \infty$, then it implies that

$$\| P_{i+1} - P_i \| \to \delta,\ i = 1,2,3.$$

That is, the triangle converges to an equilateral one. Now

$$\dot{r}_1 = \frac{1}{\sqrt{x_i^2 + y_i^2}}[x_i x_{i+1} r_{i+1} - x_i^2 r_i + y_i y_{i+1} r_{i+1} - y_i^2 r_i]$$
$$= - r_i(r_i + \delta) + \frac{r_{i+1}}{r_i + \delta}(x_i x_{i+1} + y_i y_{i+1}).$$

Choosing a Lyapunov function as

$$V = \frac{1}{2}(r_1^2 + r_2^2 + r_3^2),$$

then we have

$$\begin{aligned}\dot{V} = &- r_1^2(r_1 + \delta) + \frac{r_1 r_2}{r_1 + \delta}(x_1 x_2 + y_1 y_2) \\ &- r_2^2(r_2 + \delta) + \frac{r_2 r_3}{r_2 + \delta}(x_2 x_3 + y_2 y_3) \\ &- r_3^2(r_3 + \delta) + \frac{r_3 r_1}{r_3 + \delta}(x_3 x_1 + y_3 y_1).\end{aligned} \tag{5}$$

Note that

$$\begin{aligned} x_i x_{i+1} + y_i y_{i+1} &= \langle z_i, z_{i+1} \rangle \\ &\leqslant \| z_i \| \| z_{i+1} \| \\ &= \sqrt{x_i^2 + y_i^2}\sqrt{x_{i+1}^2 + y_{i+1}^2} \\ &= (r_i + \delta)(r_{i+1} + \delta),\ i = 1,2,3. \end{aligned} \tag{6}$$

Plugging them into (5), we have

$$\dot{V} \leqslant -(r_1^3 + r_2^3 + r_3^3 - r_1 r_2^2 - r_2 r_3^2 - r_3 r_1^2) - \delta(r_1^2 + r_2^2 + r_3^2 - r_1 r_2 - r_2 r_3 - r_3 r_1).$$

Using the inequality [9]:

$$r_1^{p_1} r_2^{p_2} \leqslant \frac{p_1}{p} r_1^p + \frac{p2}{p} r_2^p, \ p = p_1 + p_2,$$

we have $\dot{V} \leqslant 0$.

Note that from (6) $\dot{V} = 0$ if and only if

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3}, \quad (7)$$

which means P_1, P_2, P_3 lie on a straight line. An easy geometric argument shows that if (7) does not hold at $t = 0$, it can never be true for all $t > 0$.

We conclude that in the attraction-repulsion model (4), the triangle formed by three particles converge to an equilateral triangle.

5 Conclusion

Symmetry is one of the most important characteristics in real world. Many complex systems have this property. The purpose of this paper is to explore the reason for symmetry. It was found that in many cases optimizations lead to symmetries. In this paper this claim has been demonstrated by several examples in three aspects: 1) physical phenomena; 2) geometry; 3) dynamic evolution.

It is also helpful to assume certain symmetries when an optimization problem, involving several similar parts, is considered.

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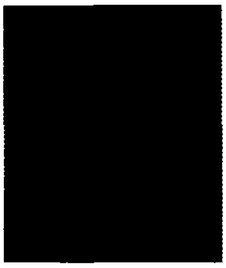

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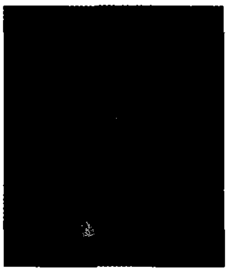

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