Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER Computers and Mathematics with Applications 48 (2004) 1847–1861
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Parameter Identification for a Class of Abstract Nonlinear Parabolic Distributed Parameter Systems

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(Received July 2003; revised and accepted July 2004)

Abstract—This paper studies the parameter identification problem of nonlinear abstract parabolic distributed parameter systems via variational method [1]. Based on the fundamental optimal control theory and the transposition method studied in [2], the existence of optimal parameter is proved, and the necessary condition for the optimal parameter is established. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Nonlinear parabolic distributed parameter system, Parameter identification, Optimal parameter, Necessary optimality condition.

1. INTRODUCTION

In recent years, there are various theoretical and numerical methods for identifying or estimating unknown parameter [3–8]. The inverse or parameter identification problem for parabolic distributed parameter systems has been studied by many researchers, see for example, [9–14]. Reference [15] proposes an approximation process for identification of nonlinearities in parabolic boundary value problem, and [16] gives a computational approach to identifying functional parameter using the gradient method. This paper will study the parameter identification problem for nonlinear parabolic distributed parameter systems involving parameter in differential operators and nonlinear terms using variational method proposed by Dautary and Lions [1].

Let Ω be an open bounded set in $\mathbf{R}^n$ with a piecewise smooth boundary $\Gamma = \partial\Omega$, q be a parameter and $Q \subset \mathbf{R}^1$ be a parameter set. We introduce two Hilbert spaces H and V with the Gelfand triple. Consider a system governed by a nonlinear parabolic evolution equation in the Hilbert space H of the form,

$$\begin{aligned} \frac{dy(t,q)}{dt} + A(t,q)\,y(t,q) &= f(t,q;y(t,q)), && \text{in } (0,T), \\ y(0,q) &= y_0, && \text{on } \Omega, \end{aligned} \tag{1.1}$$

This work is supported by National Natural Science Foundation of China (No. 60274010, No. 60334040, No. 60225003) and China Postdoctoral Science Foundation.
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0898-1221/04/$ - see front matter Typeset by AMS-TEX
doi:10.1016/j.camwa.2004.07.010

where the mapping $A : [0,T] \times Q \to \mathcal{L}(V,V')$ is a time-depending differential operator defined by some bilinear form on Hilbert space V and $f : [0,T] \times Q \times H \to H$ is a nonlinear forcing function. A and f contain unknown parameter q, which should be identified by some identification process. A powerful tool for identifying unknown parameter is the so-called output least-square estimate. The optimal control theoretical technique due to Lions [17] has shown its effectiveness in various applications to practical identification problems. We also use this method for the nonlinear system (1.1), and consider the output error criterion determined by the quadratic cost as follows,

$$J(q) = \|Cy(q) - z_d\|_{\mathcal{M}}^2, \qquad \in Q_{\text{ad}} \subset Q, \tag{1.2}$$

where $y(q)$ is a solution of (1.1), C is an observation operator, $\mathcal{M}$ is a Hilbert space (observation space), z_d is a desired value in $\mathcal{M}$. Q_{ad} is the admissible subset of Q. We shall estimate the unknown parameter q by minimizing the quadratic cost function. This is so-called output least square identification problem (OLSIP).

We study two fundamental identification problems of system (1.1) with criterion (1.2).

(i) Existence of a minimizing element $\bar{q} \in Q_{\text{ad}}$, such that,

$$\inf_{q \in Q_{\text{ad}}} J(q) = J(\bar{q}). \tag{1.3}$$

(ii) Characterization of such element $\bar{q}$.

We shall call $\bar{q}$ the optimal parameter of the system (1.1) with respect to (1.2).

The purpose of this paper is to prove the existence and to provide necessary conditions of the optimal parameter for the nonlinear parabolic system (1.1) with respect to (1.2). The content of this paper is as follows. The notations, definitions and auxiliary theorem will be given in Section 2. In Section 3, the strong continuity of $y(q)$ with respect to q, and the existence of the optimal parameter $\bar{q}$ will be proved. Then, a necessary optimality condition for optimal parameter $\bar{q}$ will be established.

2. PRELIMINARIES

First of all, we explain the notations used in this paper. Let H and V be two real Hilbert spaces with norm denoted by $|\cdot|_H$ and $\|\cdot\|_V$, respectively. The symbol $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$ denote the inner product on H and V, respectively. V' denotes the dual space of V and $\langle\cdot,\cdot\rangle_{V',V}$ denotes the dual pairing between V' and V. Assume that (V,H,V') is a Gelfand triple space with $V \hookrightarrow H \equiv H' \hookrightarrow V'$, which means that the embedding $V \hookrightarrow H$ is continuous and V is dense in H. Let $0 < T < \infty$, $\mathbf{R}^1 = (-\infty,\infty)$ and $\mathbf{R}^+ = [0,\infty)$. Let Q be a set of $\mathbf{R}^1$ and Q_{ad} (respectively Q_{bd}) be a convex (respectively bounded) subset of Q.

For each $q \in Q$ and $t \in [0,T]$, we consider a bilinear form $a(t,q;\phi,\varphi)$ on $V \times V$, satisfying

(i) $a(t,q;\phi,\psi) = a(t,q;\psi,\phi)$, for all $\phi,\psi \in V, t \in [0,T]$;

(ii) there exists a $c(q) > 0$, such that $|a(t,q;\phi,\psi)| \le c(q)\|\phi\|_V \|\psi\|_V$, for all $\phi,\psi \in V$ and $t \in [0,T]$;

(iii) there exist $\alpha(q) > 0$ and $\lambda(q) \in \mathbf{R}$, such that $a(t,q;\phi,\phi) + \lambda(q)|\phi|_H^2 \ge \alpha(q)\|\phi\|_V^2$, for all $\phi \in V$ and $t \in [0,T]$.

We suppose that for each Q_{bd}, there exist positive numbers c,λ,α such that,

$$c(q) \le c, \qquad \lambda(q) \le \lambda, \qquad \alpha(q) \ge \alpha, \qquad \text{for all } q \in Q_{\text{bd}}. \tag{2.1}$$

Then, we can define an operator $A(t,q) \in \mathcal{L}(V,V')$, for $t \in [0,T]$ via the relation,

$$a(t,q;\phi,\varphi) = \langle A(t,q)\phi,\varphi\rangle_{V',V}, \qquad \text{for all } \phi,\varphi \in V, \tag{2.2}$$

where $\mathcal{L}(V, V')$ is the Banach space of all bounded linear operators of V to V' with the uniform operator's topology. We often use g' to express $\frac{dg}{dt}$, the time derivative of g.

We define a Hilbert space $W(0,T)$, which will be a solution space, by

$$W(0,T) = \{g \mid g \in L^2(0,T;V),\ g' \in L^2(0,T;V')\}.$$

The inner product and the induced norm in $W(0,T)$ are defined by

$$(g_1, g_2)_{W(0,T)} = \int_0^T \{(g_1(t), g_2(t))_V + (g_1'(t), g_2'(t))_{V'}\}\, dt,$$

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;V)} + \|g'\|^2_{L^2(0,T;V')}\right)^{1/2},$$

respectively.

For each fixed $q \in Q$, we consider the Cauchy problem for the nonlinear parabolic evolution equation

$$\begin{gathered} \frac{dy}{dt} + A(t,q)\,y = f(t,q;y), \qquad \text{in } [0,T], \\ y(0) = y_0, \end{gathered} \tag{2.3}$$

where $f : [0,T] \times Q \times H \to H$ is a nonlinear forcing function.

We impose the following assumptions to the nonlinear term f in (2.3):

(A1) for each $(q, y) \in Q \times H$, the mapping $t \to f(t,q;y)$ is strongly measurable in H;

(A2) for each $q \in Q$, there exists a $\beta(\cdot, q) \in L^2(0,T;\mathbf{R}^+)$, such that, for all $y, z \in H$

$$|f(t,q;y) - f(t,q;z)| \le \beta(t,q)\,|y - z|, \qquad \text{a.e. in } [0,T];$$

(A3) For each $q \in Q$, there exists a $\gamma(\cdot, q) \in L^2(0,T;\mathbf{R}^+)$, such that,

$$|f(t,q;0)| \le \gamma(t,q), \qquad \text{a.e. in } [0,T].$$

We suppose that, for each Q_{bd}, there exist functions $\beta_1 \in L^2(0,T;\mathbf{R}^+)$ and $\gamma_1 \in L^2(0,T;\mathbf{R}^+)$ such that,

$$\beta(t,q) \le \beta_1(t), \qquad \gamma(t,q) \le \gamma_1(t), \qquad \forall\, q \in Q_{\mathrm{bd}}, \quad t \in \mathbf{R}^+. \tag{2.4}$$

Now, we give the definition of weak solution of (2.3) due to Dautray and Lions [1].

DEFINITION 1. *A function $y = y(q)$ is said to be a weak solution of (2.3), if $y \in W(0,T)$, satisfies*

$$\begin{gathered} \langle y'(\cdot,q), v\rangle_{V',V} + a(\cdot,q;y(\cdot,q),v) = (f(\cdot,q;y(\cdot,q)), v)_H, \\ \text{for all } v \in V, \text{ in the sense of } \mathcal{D}'(0,T), \\ y(0,q) = y_0 \in H. \end{gathered} \tag{2.5}$$

Here, $\mathcal{D}'(0,T)$ denotes the space of distributions on $(0,T)$.

The following theorem about existence, uniqueness and regularity of the solution of (2.5) can be proved by using the Galerkin method as in [18].

THEOREM 1. *Assume that $a(t,q;\phi,\psi)$ satisfies (i)–(iii). If $y_0 \in H$ and $f(t,q;y)$ satisfies Assumptions (A1)–(A3), then problem (2.3) has a unique weak solution y in $W(0,T)$. Furthermore, $y \in C([0,T];H)$ has an estimate*

$$|y(t,q)|^2 + \int_0^t \|y(s,q)\|^2\, ds \le C(t,q)\left(|y_0|^2 + |f(t,q;0)|^2\right), \qquad \forall\, t \in [0,T], \tag{2.6}$$

where $C(t,q)$ is a positive constant, depending on q and t but independent of y_0 and f.

REMARK 1. From conditions (2.1) and (2.4), the constant $C(t,q)$ in (2.6) has a finite upper bound $C(t) \in L^\infty(0,T;\mathbf{R}^+)$ for each Q_{bd}, i.e.,

$$C(t,q) \le C(t) < \infty, \qquad \forall\, q \in Q_{\mathrm{bd}}.$$

Hence, (2.6) can be rewritten as

$$|y(t,q)|^2 + \int_0^t \|y(s,q)\|^2\, ds \le C(t)\left(|y_0|^2 + \|\gamma\|^2_{L^2(0,T;\mathbf{R}^+)}\right). \tag{2.6}$$

3. PROBLEM OF IDENTIFICATION

For each parameter $q \in Q$, we consider the following nonlinear Cauchy problem involving q in differential operator and nonlinear forcing function

$$\begin{aligned} y'(t,q) + A(t,q)\, y(t,q) &= f(t,q;y(t,q)), \quad \text{in } (0,T), \\ y(0;q) &= y_0 \in H, \end{aligned} \tag{3.1}$$

where $A(t,q)$ is a differential operator, and $f(t,q;y)$ is a nonlinear forcing function satisfying the assumptions (A1)–(A3). By virtue of Theorem 1, for each parameter $q \in Q$, there is a unique weak solution $y = y(t,q)$ to (3.1). Therefore, we have a well-defined mapping $q \to y(q)$ of Q into $W(0,T)$. We shall call $y(t,q)$ the state of system (3.1).

3.1. Strong Continuity of the Solution Mapping on Parameter

In this section, we shall establish the strong continuity of the mapping $q \to y(t,q)$. For this purpose, we require the continuity of $A(t,q)$ and $f(t,q;y)$ on the parameter q. More precisely, we make the following assumptions.

(B1) There exists $k_1 \in C([0,T] \times \mathbf{R}^+)$ with $k_1(t,0) \equiv 0$ such that,

$$|a(t,q;\phi,\psi) - a(t,p;\phi,\psi)| \le k_1\left(t, \|q-p\|_Q\right) \|\phi\|_V \|\psi\|_V, \quad \forall\, q,p \in Q,\ \forall\, \phi,\psi \in V.$$

(B2) There exists $k_2 \in C([0,T] \times \mathbf{R}^+ \times H)$ with $k_2(t,0,y) \equiv 0$ such that,

$$|f(t,q;y) - f(t,p;y)| \le k_2\left(t, \|q-p\|_Q, |y|\right), \quad \forall\, q,p \in Q,\ y \in H.$$

THEOREM 2. *Assume that (i)–(iii) and (B1),(B2) hold. Then, the mapping $q \to y(q) : Q \to W(0,T)$ is strongly continuous.*

PROOF. For any fixed parameter q, let $\{q_n\} \subset Q$ be a sequence, such that $\|q_n - q\|_Q \to 0$, as $n \to \infty$. Let $y_n = y(q_n)$ be the weak solution of

$$\begin{aligned} y_n' + A(t,q_n)\, y_n &= f(t,q_n;y_n), \quad t \in (0,T), \\ y_n(0) &= y_0 \in H. \end{aligned} \tag{3.2}$$

Since the set $Q_{\text{bd}} = \{q_n \mid n \ge 1\} \cup \{q\}$ is bounded in Q, it follows from (2.7) that

$$|y_n(t)|^2 + \int_0^t \|y_n(s)\|^2\, ds \le C(t)\left(|y_0|^2 + \|\gamma\|^2_{L^2(0,T;\mathbf{R}^+)}\right), \qquad \forall\, t \in [0,T]. \tag{3.3}$$

Hence, $\{y_n\}$ is bounded in $L^\infty(0,T;H)$. Also, the boundedness of $\{f(t,q_n;y_n)\}$ in $L^2(0,T;H)$ follows easily from

$$|f(t,q_n;y_n)| \le |f(t,q_n;0)| + \beta(t)\,|y_n| \le \gamma(t) + \beta(t)\,|y_n|,$$

where $\gamma_1(t)$ and $\beta_1(t)$ are square integrable functions corresponding to $Q_{\text{bd}} = \{q_n \mid n \ge 1\} \cup \{q\}$. Since $\{A(t,q_n)y_n\}$ is bounded in $L^2(0,T;V')$, $\{y_n'\}$ is bounded in $L^2(0,T;V')$ by (ii). Hence, $\{y_n\}$ is bounded in $W(0,T)$, and then, we can extract a sub-sequence, written by $\{y_n\}$ still, and find a $z \in W(0,T)$ with $z(0) = y_0$ and a $Y \in L^2(0,T;H)$, such that,

$$\begin{aligned} y_n &\to z, && \text{weakly in } L^2(0,T;V), \\ y_n &\to z, && \text{weakly star in } L^\infty(0,T;H), \\ y_n' &\to z', && \text{weakly in } L^2(0,T;V'), \\ f(t,q_n;y_n) &\to Y, && \text{weakly in } L^2(0,T;H). \end{aligned} \tag{3.4}$$

Also by (3.4), we know that, for all fixed $t \in [0,T]$,

$$\begin{aligned} y_n(t) &\to z(t), && \text{weakly in } V, \\ y_n'(t) &\to z'(t), && \text{weakly in } V'. \end{aligned} \tag{3.5}$$

The equation (3.2) is rewritten as

$$y_n' + A(t,q)\,y_n = (A(t,q) - A(t,q_n))\,y_n + f(t,q_n;y_n). \tag{3.6}$$

Let $\phi \in L^2(0,T;V)$ be fixed. Then, we multiply both sides of the above equality (3.6) by ϕ and integrate over $[0,T]$. By the definition of weak solution, we have

$$\begin{aligned} &\int_0^T \langle y_n' + A(t,q)\,y_n, \phi\rangle_{V',V}\,dt \\ &= \int_0^T \langle (A(t,q) - A(t,q_n))\,y_n, \phi\rangle_{V',V}\,dt + \int_0^t (f(t,q_n;y_n), \phi)\,dt. \end{aligned} \tag{3.7}$$

Now, in terms of (B1), we deduce

$$|\langle (A(t,q) - A(t,q_n))\,\phi, y_n\rangle_{V',V}| \le k_1\left(t, \|q_n - q\|_Q\right)\|\phi\|\,\|y_n\|.$$

Since $\{y_n\}$ is bounded in $L^2(0,T;V)$ and k_1 is continuous, the above inequality implies, by the Lebesgue dominated convergence theorem, that

$$\int_0^T |\langle (A(t,q) - A(t,q_n))\,\phi, y_n\rangle_{V',V}|\,dt \to 0, \qquad \text{as } n \to \infty.$$

Hence, in the equality (3.7) letting $n \to \infty$, and using the weak convergence in (3.4), we have

$$\int_0^T \langle z' + A(t,q)\,z, \phi\rangle_{V',V}\,dt = \int_0^t (Y,\phi)\,dt, \qquad \phi \in L^2(0,T;V). \tag{3.8}$$

This implies that $z = z(t,q)$ is a unique weak solution of the linear equation

$$\begin{aligned} z'(t,q) + A(t,q)\,z(t,q) &= Y, \qquad \text{in } (0,T), \\ z(0,q) &= y_0 \in H. \end{aligned} \tag{3.9}$$

We shall show $Y(t) = f(t,q;z)$, and then, by the uniqueness of the solution of (3.2), we obtain $z = y(q)$. For this, we shall prove the strong convergence of y_n to z. From the energy equality for y_n and z, we have

$$\begin{aligned} &|y_n(t)|^2 + 2\int_0^t a(s,q_n;y_n(s),y_n(s))\,ds \\ &= |y_0|^2 + 2\int_0^t (f(s,q_n;y_n(s)), y_n(s))\,ds, \qquad \forall t \in [0,T], \end{aligned} \tag{3.10}$$

and

$$|z(t)|^2 + 2\int_0^t a(s,q;z(s),z(s))\,ds = |y_0|^2 + 2\int_0^t (Y(s), z(s))\,ds, \qquad \forall t \in [0,T]. \tag{3.11}$$

For the simplicity of notations, the arguments t and s of function z in the following calculations are omitted. Adding (3.10) to (3.11), we have

$$\begin{aligned} &|y_n - z|^2 + 2\int_0^t a(s,q_n;y_n - z, y_n - z)\,ds \\ &= 2\sum_{i=0}^{3} Y_n^i(t) + 2\int_0^t (f(s,q_n;y_n) - f(s,q_n;z), y_n - z)\,ds, \end{aligned} \tag{3.12}$$

where

$$
\begin{aligned}
Y_n^0(t) &= -(y_n, z)_H, \\
Y_n^1(t) &= -2\int_0^t (a(s,q_n;y_n,z) - a(s,q;y_n,z))\,ds - 2\int_0^t a(s,q;y_n,z)\,ds, \\
Y_n^2(t) &= \int_0^t (f(s,q_n;z) - f(s,q;z), y_n - z)\,ds + \int_0^t (f(s,q;z), y_n - z)\,ds \\
&\quad + \int_0^t (f(s,q_n;y_n) - Y, z)\,ds, \\
Y_n^3(t) &= \int_0^t (a(s,q_n;z,z) - a(s,q;z,z))\,ds.
\end{aligned}
$$

Note that,

$$\left|2\int_0^t (f(s,q_n;y_n) - f(s,q_n;z), y_n - z)\,ds\right| \le 2\int_0^t \beta(s)\,|y_n - z)|^2\,ds.$$

We set $Y_n(t) = \sum_{i=0}^3 Y_n^i(t)$ in (3.12). By using (i)–(iii), we have

$$|y_n - z|^2 + 2\alpha\int_0^t \|y_n - z\|^2\,ds \le 2Y_n(t) + 2\int_0^t (\beta(s) + \lambda)\,|y_n - z|^2\,ds, \tag{3.13}$$

where α, λ are positive constants independent of q_n. Denote $\mu = \min\{1, 2\alpha\}$, and set

$$
\begin{aligned}
&\Phi_n(t) = |y_n - z|^2 + \int_0^t \|y_n - z\|^2\,ds, \\
&Z_n(t) = 2\mu^{-1}Y_n(t), \\
&h(t) = 2\mu^{-1}(\beta(t) + \lambda),
\end{aligned}
$$

then, the inequality (3.13) implies that,

$$\Phi_n(t) \le Z_n(t) + \int_0^t h(s)\,\Phi_n(s)\,ds.$$

Since $Z_n(t)$ is continuous, we can apply the extended Bellman-Gronwall inequality (cf. [19]) to get

$$\Phi_n(t) \le Z_n(t) + \int_0^t \exp\left(\int_s^t h(\tau)\,d\tau\right) h(s)\,Z_n(s)\,ds. \tag{3.14}$$

We claim that $\lim_{n\to\infty}\Phi_n(t) = 0$, for each $t \in [0,T]$. Let

$$K(t,s) = \exp\left(\int_s^t h(\tau)\,d\tau\right) h(s), \qquad M_n(t) = \int_0^t K(t,s)\,Z_n(s)\,ds. \tag{3.15}$$

Then,

$$\Phi_n(t) \le Z_n(t) + \int_0^t K(t,s)\,Z_n(s)\,ds.$$

Moreover, it is easy to see that

$$|K(t,s)| \le \exp\left(\|h\|_{L^1(0,T;\mathbf{R}^+)}\right) h(s),$$

and $M_n(t)$ is uniformly bounded on $[0,T]$. In order to verify $Z_n(t) \to 0$, as $n \to \infty$, it is sufficient to prove

$$\lim_{n\to\infty} Y_n(t) = 0, \quad \forall\, t \in [0,T]. \tag{3.16}$$

By (B1), Lebesgue dominated convergence theorem and boundedness of $\{y_n\} \in L^2(0,T;H)$, we have

$$\begin{aligned}
&\left|\int_0^t a\left(s,q_n;y_n,z\right) - a\left(s,q;y_n,z\right)\,ds\right| \\
&\leq \int_0^t k_1\left(s,|q_n-q|_Q\right)|y_n|\,|z|\,ds \\
&\leq \left(\int_0^t k_1^2\left(s,\|q_n-q\|_Q\right)|z|^2\,ds\right)^{1/2}\left(\int_0^t |y_n|^2\,ds\right)^{1/2} \to 0,
\end{aligned}$$

as $n \to \infty$. Then, we obtain

$$\lim_{n\to\infty} Y_n^1(t) = -2\int_0^t a\left(s,q;z,z\right)\,ds. \tag{3.17}$$

Similarly, we have, from the assumptions (B1), (B2), and (3.4), that

$$Y_n^0(t) \to -|z|^2, \quad Y_n^2(t) \to 2\int_0^t \langle Y,z\rangle_{V,V'}\,ds, \qquad Y_n^3(t) \to 0, \qquad \text{as } n\to\infty. \tag{3.18}$$

Therefore by (3.11) with $|y_0| = 0$, (3.17) and (3.18), we have the desired result (3.16). This proves the claim,

$$\lim_{n\to\infty} \Phi_n(t) = 0, \quad \forall\, t \in [0,T].$$

Hence,

$$y_n \to z, \qquad \text{strongly in } C\left([0,T];V\right) \text{ and } L^2\left(0,T;V\right). \tag{3.19}$$

$$y_n \to z, \qquad \text{strongly in } C\left([0,T];H\right) \text{ and } L^\infty\left(0,T;H\right). \tag{3.20}$$

Now, it is ready to verify $Y(t) = f(t,q;z(t))$ in H, for a.e. $t \in [0,T]$. Applying the above convergence and (B2), we get the inequality

$$\begin{aligned}
|f(t,q_n;y_n) - f(t,q;z)| &\leq |f(t,q_n;y_n) - f(t,q_n;z)| + |f(t,q_n;z) - f(t,q;z)| \\
&\leq \beta(t)\,|y_n - z| + k_2\left(t,\|q_n-q\|_Q,|z|\right).
\end{aligned} \tag{3.21}$$

Finally, taking the difference of (3.2) and (3.9), we have

$$(y_n - z)' = \left(A(t,q_n) - A(t,q)\right)y + A(t,q)\left(y_n - z\right) + f\left(t,q_n;y_n\right) - f\left(t,q;z\right).$$

Note that by (3.21) and (B2), the last term in the right-hand side of the above equality strongly converges to 0 in $L^2(0,T;H)$. Then, it follows from (B1), (3.19), and (3.21) that

$$y_n' \to z', \qquad \text{strongly in } L^2\left(0,T;V'\right).$$

Since any sequence $\{q_n\}$ converging to q in Q has a sub-sequence q_{n_k}, such that $y(q_{n_k}) \to y(q)$ in $W(0,T)$, we conclude that the mapping $q \to y(q)$ is strongly continuous in $W(0,T)$.

3.2. Existence of Optimal Parameter

We now consider the problem of existence of optimal parameter. The performance criterion is given by

$$J(q) = \|\mathcal{C}y(q) - z_d\|_{\mathcal{M}}^2, \qquad \text{for } q \in Q, \tag{3.22}$$

where $\mathcal{M}$ is a Hilbert space of observations, $\mathcal{C} \in \mathcal{L}(W(0,T),\mathcal{M})$ is an observation operator and z_d is a desired value belonging to $\mathcal{M}$. Our goal is to find an optimal element $\bar{q} \in Q_{\text{ad}}$, such that

$$J(\bar{q}) = \min_{q\in Q_{\text{ad}}} J(q), \tag{3.23}$$

and to derive a necessary condition for optimal parameter $\bar{q}$. We call $\bar{q}$, the optimal parameter and $y = y(\bar{q})$, the optimal state.

THEOREM 3. *Assume that (B1),(B2) hold. If Q_{ad} is compact and convex in Q, then, there exists at least one optimal parameter $\bar{q} \in Q_{\text{ad}}$.*

PROOF. Since Q_{ad} is a nonempty convex subset of Q, there is a minimizing sequence $\{q_n\}$, such that,

$$\lim_{n\to\infty} J(q_n) = \inf_{q\in Q_{\text{ad}}} J(q).$$

Since Q_{ad} is compact, there exists a sub-sequence $\{q_{n_k}\}$ and a $\bar{q} \in Q_{\text{ad}}$ such that,

$$\|q_{n_k} - \bar{q}\|_Q \to 0.$$

Hence, by Theorem 1,

$$y(q_{n_k}) \to y(\bar{q}) \text{ in } W(0,T).$$

It follows from (3.22) that,

$$J(\bar{q}) = J\left(\lim_{k\to\infty} q_{n_k}\right) = \lim_{k\to\infty} J(q_{n_k}) = \inf_{q\in Q_{\text{ad}}} J(q).$$

This proves that $\bar{q}$ is an optimal parameter.

3.3. Necessary Optimality Condition

We now consider the necessary condition for the optimal parameter $\bar{q}$. One classical method to obtain the necessary condition for $\bar{q}$ is to calculate the first variation of $J(q)$ around $\bar{q}$. If $y(q)$ is Gâteaux differentiable at $\bar{q} \in Q_{\text{ad}}$ and $y'(\bar{q})$ is its Gâteaux derivative at $q = \bar{q}$, then $J(q)$ is Gâteaux differentiable at $q = \bar{q}$, and the necessary condition for the optimal parameter $\bar{q}$ is characterized by the following variational inequality,

$$J'(\bar{q})(q - \bar{q}) \geq 0, \qquad \forall q \in Q_{\text{ad}},$$

where $J'(\bar{q})$ denotes the Gâteaux derivative of $J(q)$. Therefore, we consider Gâteaux differentiability of $y(q)$ at $\bar{q}$. In the following, for a linear operator L, L^* denotes the adjoint of L.

In this section, we pose following assumptions.

(C1) For each $y \in H$, $f(t,q;y)$ is Gâteaux differentiable with respect to $q \in Q_{\text{ad}}$ for a.e., $t \in [0,T]$, and for any $q \in Q_{\text{ad}}$, $f(t,q;y)$ is Fréchet differentiable with respect to $y \in H$ for a.e., $t \in [0,T]$. The Gâteaux derivative $f'_q(t,q;y)$ and the Fréchet derivative $f_y(t,q;y)$ are continuous on $Q_{\text{ad}} \times H$ for a.e., $t \in [0,T]$. Moreover, for some bounded subset H_{bd} of H, there are $\tilde{\beta}_1(\cdot), \tilde{\beta}_2(\cdot) \in L^2(0,T;\mathbf{R}^+)$, depending on H_{bd}, such that

$$\left\|f'_q(t,q;y)\right\|_{\mathcal{L}(Q,H)} \leq \tilde{\beta}_1(t), \qquad \forall (q,y) \in Q_{\text{ad}} \times H_{\text{bd}}, \quad \text{for a.e., } t \in [0,T].$$
$$\left\|f^*_y(t,q;y)\right\|_{\mathcal{L}(H)} \leq \tilde{\beta}_2(t), \qquad \forall (q,y) \in Q_{\text{ad}} \times H_{\text{bd}}, \quad \text{for a.e., } t \in [0,T].$$

(C2) For any $\phi, \psi \in V$, $a(t,q;\phi,\psi)$ is Gâteaux differentiable with respect to $q \in Q_{\text{ad}}$, for all $t \in [0,T]$, and there exists $\tilde{\gamma} > 0$, such that Gâteaux derivative $a'_q(t,q;\phi,\psi)$ satisfies

$$\left\|a'_q(t,q;\phi,\psi)\right\|_{\mathcal{L}(Q,\mathbf{R})} \leq \tilde{\gamma}\|\phi\|\;\|\psi\| \qquad \forall (t,q) \in [0,T] \times Q_{\text{ad}}.$$

The purpose of this section is to describe the optimality condition (3.24) in terms of proper adjoint system.

In order to prove the Gâteaux differentiability of $q \mapsto (y(T,q), y(q))$ in the space $H \times L^2(0,T;V)$, we use the transposition method. We need some preliminaries on the adjoint equations which are related to the first variation of $(y(T;q), y(q))$ with respect to q. Furthermore, let $y'(\bar{q}, q - \bar{q})$

denotes the Gâteaux differential of $y(q)$ with respect to q at $\bar{q}$ in the direction $q-\bar{q}$, we shall derive the equation of $y'(\bar{q}, q-\bar{q})$.

For any $q \in Q_{\text{ad}}$ and $\lambda \in [0,1]$, suppose that there is an operator $L(q,\lambda;t) \in \mathcal{L}(H)$ for a.e., $t \in [0,T]$. Consider the following terminal value problem of linear evolution equation,

$$\begin{aligned} -\phi' + A^*(t,q)\,\phi + L(q,\lambda;t)\,\phi &= g, \qquad t \in (0,T), \\ \phi(T) &= \phi_T, \end{aligned} \tag{3.25}$$

where $\phi_T \in H, g \in L^2(0,T;H)$. By Theorem 1, if we consider reversed time flow $t \to T-t$, the linear problem (3.25) has a unique weak solution $\phi = \phi(q,\lambda;\phi_T,g) \in W(0,T)$. Furthermore, if there exists a $\beta(\cdot) \in L^2(0,T;\mathbf{R}^+)$ such that,

$$\|L(q,\lambda;t)\|_{\mathcal{L}(H)} \le \beta(t), \qquad \text{for, a.e., } t \in [0,T], \quad \forall (q,\lambda) \in Q_{\text{ad}} \times [0,1], \tag{3.26}$$

then, we have an estimate

$$\max_{t\in[0,T]} \|\phi\| \le c\left(|\phi_T| + \|g\|_{L^2(0,T;H)}\right), \tag{3.27}$$

where c depends on q and λ but is independent of ϕ_T and g. This assures that $\phi_T \in H, g \in L^2(0,T;H)$ and $L(q,\lambda;t) \in L^2(0,T;\mathcal{L}(H))$.

Let $X[q,\lambda]$ be the set of all weak solutions of (3.25), i.e.,

$$X[q,\lambda] = \left\{\phi \mid \phi = \phi(q,\lambda;\phi_T,g) \in W(0,T),\ \phi_T \in H,\ g \in L^2(0,T;H)\right\}. \tag{3.28}$$

Since the equation (3.25) with $L(q,\lambda;t)$ is linear, we can define an inner product on $X[q,\lambda]$ by

$$(\phi,\psi)_{X[q,\lambda]} = (\phi_T,\psi_T)_H + (g,h)_{L^2(0,T;H)},$$

for $\phi = \phi(q,\lambda;\phi_T,g), \psi = \phi(q,\lambda;\psi_T,h)$. It is easily verified that $(X[q,\lambda], (\cdot,\cdot)_{X[q,\lambda]})$ is a Hilbert space, and the map $\phi = \phi(q,\lambda;\phi_T,g) \to (\phi_T,g)$: $X[q,\lambda]$ onto $H \times L^2(0,T;H)$ is an isomorphism. Define the operator

$$\Phi[q,\lambda;t](\phi) = -\phi' + A^*(t,q)\,\phi + L(q,\lambda;t)\,\phi. \tag{3.29}$$

Using the method of transposition due to Lions and Magenes [2], for a bounded linear functional l defined on $X[q,\lambda]$, there is a unique solution $\zeta \in L^2(0,T;H)$ such that,

$$(\zeta(T),\phi(T))_H + \int_0^T (\zeta(t), \Phi[q,\lambda;t]\,\phi(t))\,dt = l(\phi), \qquad \text{for all } \phi \in X[q,\lambda]. \tag{3.30}$$

Particularly, for $(q,\lambda) = (\bar{q},0) \in Q_{\text{ad}} \times [0,1]$, we define $L(\bar{q},0;t) = -f_y^*(t,\bar{q},y(\bar{q})) \in \mathcal{L}(H)$, and denote by $\Phi[\bar{q},0;t]$ the corresponding operator in (3.29), by $X[\bar{q},0]$ the solution space of this $(\bar{q},0)$.

THEOREM 4. *Assume that (B1),(B2) and (C1),(C2) hold. Then, the map $q \mapsto y(q)$ of Q into $W(0,T)$ is Gâteaux differentiable at $\bar{q}$, and the Gâteaux differential of $y(q)$ at $\bar{q}$ in the direction $q-\bar{q} \in Q$, denoted by $z = y'(\bar{q})(q-\bar{q})$, is the unique solution of*

$$\begin{aligned} &(z(T),\phi(T))_H + \int_0^T \langle z, -\phi' + A^*(t,q)\,\phi - f_y^*(t,\bar{q};y(\bar{q}))\,\phi\rangle_{V,V'}\,dt \\ &= -\int_0^T a_q'(t,\bar{q};y(\bar{q}),\phi)(q-\bar{q})\,dt + \int_0^T \left(f_q'(t,\bar{q};y(\bar{q}))(q-\bar{q}),\phi\right)\,dt, \\ &\qquad z(0) = 0. \end{aligned} \tag{3.31}$$

for all $\phi \in W(0,T)$.

PROOF. For simplicity, we omit the variable t. We now show Gâteaux differentiability of mapping $q \to y(q)$ at $\bar{q}$ in the direction $q - \bar{q}, q \in Q_{\text{ad}}$. For this, set $q_\lambda = \bar{q} + \lambda(q - \bar{q}), \lambda \in [0,1]$, then $q_\lambda \in Q_{\text{ad}}$ because of the convexity of Q_{ad} and $\|q_\lambda - \bar{q}\|_Q = \lambda\|q - \bar{q}\|_Q \to 0$ as $\lambda \to 0$. Since $\{q_\lambda\}_{\lambda\in[0,1]} \subset Q_{\text{ad}}$ and Q_{ad} is bounded, by Theorem 2 and (B1)–(B2), we have

$$y(q_\lambda) \to y(\bar{q}), \qquad \text{strongly in } W(0,T), \qquad \text{as } \lambda \to 0. \tag{3.32}$$

By (3.20), we also have

$$y(q_\lambda) \to y(\bar{q}), \qquad \text{strongly in } C(0,T;H), \qquad \text{as } \lambda \to 0. \tag{3.33}$$

We set $y_\lambda = y(q_\lambda) - \bar{y}$ for $\lambda \in [0,1]$ and $\bar{y} = y(\bar{q}) \in W(0,T) \cap C(0,T;V) \cap C(0,T;H)$. As in Section 3, we have the uniform boundedness

$$\sup\left\{|y_\lambda|^2 + \int_0^t \|y_\lambda\|^2\, ds \mid (t,q,\lambda) \in [0,T] \times Q_{\text{ad}} \times [0,1]\right\} < \infty. \tag{3.34}$$

For $\lambda \in [0,1]$, y_λ satisfies

$$\begin{aligned} y_\lambda' + A(q_\lambda)\, y_\lambda &= -\left(A(q_\lambda) - A(\bar{q})\right)\bar{y} + f(q_\lambda; y_\lambda) - f(\bar{q};\bar{y}), \qquad \text{in } (0,T), \\ y_\lambda(0) &= 0 \in H. \end{aligned} \tag{3.35}$$

Divide (3.35) by λ and set $z_\lambda = \lambda^{-1} y_\lambda$. Then, z_λ satisfies

$$\begin{aligned} & z_\lambda' + A(q_\lambda)\, z_\lambda - \int_0^1 f_y(t, q_\lambda; \theta y_\lambda + (1-\theta)\bar{y})\, d\theta\, z_\lambda \\ &= -\frac{A(q_\lambda) - A(q)}{\lambda}\bar{y} + \frac{f(q_\lambda;\bar{y}) - f(\bar{q};\bar{y})}{\lambda}, \qquad \text{in } (0,T), \\ & z_\lambda(0) = 0 \in H, \end{aligned} \tag{3.36}$$

in weak sense. For all $(q,\lambda) \in Q_{\text{ad}} \times [0,1]$, we set

$$L(q,\lambda;t) = -\int_0^1 f_y^*(t, q_\lambda; \theta y_\lambda + (1-\theta)\bar{y})\, d\theta.$$

Note that $L(\bar{q},0;t) = -f_y^*(t,\bar{q},y(\bar{q}))$, for all $q \in Q_{\text{ad}}$, because $q_\lambda \in Q_{\text{ad}}$ and

$$\sup\left\{|\theta y_\lambda(t) + (1-\theta)\bar{y}(t)| : (t,\theta,q,\lambda) \in [0,T] \times [0,1] \times Q_{\text{ad}} \times [0,1]\right\} < \infty.$$

Due to (3.34), it follows from (C1) that

$$\|L(q,\lambda;t)\|_{\mathcal{L}(H)} \le \int_0^1 \tilde{\beta}_2(t)\, d\theta = \tilde{\beta}_2(t), \qquad \text{a.e., } t, \text{ for all } (q,\lambda) \in Q_{\text{ad}} \times [0,1], \tag{3.37}$$

so that $L(q,\lambda;\cdot) \in L^2(0,T;\mathcal{L}(H))$. For each $(q,\lambda) \in Q_{\text{ad}} \times [0,1]$, since (3.37) implies (3.26), $\phi_T \in H$ and $g \in L^2(0,T;H)$, there is a unique weak solution $\phi = \phi(q,\lambda;\phi_T,g) \in W(0,T)$, satisfying,

$$\begin{aligned} \Phi[q,\lambda;t] = g, \qquad & L(q,\lambda;t) = -\int_0^1 f_y^*(t,q_\lambda;\theta y_\lambda + (1-\theta)\bar{y})\, d\theta, \\ & \phi(T) = \phi_T. \end{aligned} \tag{3.38}$$

Furthermore, we have the estimate (3.27) of ϕ which is independent of (q,λ).

Multiplying both sides of (3.36) by $\phi \in W(0,T)$ with $\phi(T) = \phi_T$, we have a weak form

$$
\begin{aligned}
&(z_\lambda(T), \phi(T))_H + \int_0^t (z_\lambda(q), \Phi[q,\lambda;t](\phi))\, dt \\
&= -\int_0^t \frac{a(t, q_\lambda; y_\lambda, \phi) - a(t, \bar{q}; y_\lambda, \phi)}{\lambda}\, dt + \int_0^t \left(\frac{f(t, q_\lambda; y(\bar{q})) - f(t, \bar{q}; y(\bar{q}))}{\lambda}, \phi \right) dt \\
&= I_1(\phi) + I_2(\phi).
\end{aligned} \tag{3.39}
$$

Let us estimate I_1, I_2 by using (C1),(C2). There exist $\theta_1, \theta_2 \in [0,1]$ such that,

$$
\begin{aligned}
|I_1(\phi)| &= \left| \int_0^T a'_q(t, \bar{q} + \theta_1 \lambda (q - \bar{q}); y_\lambda, \phi)(q - \bar{q})\, dt \right| \\
&\le \tilde{\gamma} \|q - \bar{q}\|_Q \max_{t\in[0,T]} \|y_\lambda\| \max_{t\in[0,T]} \|\phi\| \le c' \max_{t\in[0,T]} \|\phi\|.
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2(\phi)| &= \left| \int_0^T (f'_q(t, \bar{q} + \theta_2 \lambda (q - \bar{q}); \bar{y})(q - \bar{q}), \phi)\, dt \right| \\
&\le \left\| \tilde{\beta}_1 \right\|_{L^2(0,T;\mathbf{R}^+)} \tilde{\gamma} \|q - \bar{q}\|_Q \max_{t\in[0,T]} \|\phi\| \le c'' \max_{t\in[0,T]} \|\phi\|.
\end{aligned}
$$

Hence,

$$
|I_1(\phi)| + |I_2(\phi)| \le (c' + c'') \max_{t\in[0,T]} \|\phi\|,
$$

where c', c'' are positive constants independent of $q \in Q_{\text{ad}}$. Then, we can easily show that z_λ is bounded in $L^\infty(0,T;H)$ by taking $\phi_T = 0, g = z_\lambda$ in (3.38), also z_λ is bounded in $L^2(0,T;V)$ by taking $\phi_T = 0, g = A(t,q_\lambda)z_\lambda$ in (3.38). Further, taking $\phi_T = z_\lambda(T), g(t) = 0$, then, we can easily see that $\{z_\lambda(T)\}$ is bounded in H. Therefore, we can extract a subsequence $\{z_\lambda\}$, denoted by itself, and find a $z \in L^2(0,T;V), z(T) \in H$ such that,

$$
z_\lambda \to z, \qquad \text{weakly in } L^2(0,T;V). \tag{3.40}
$$

$$
z_\lambda(T) \to z(T), \qquad \text{weakly in } H. \tag{3.41}
$$

Fix $\phi \in X[\bar{q}, 0] \subset W(0,T)$ with $L(\bar{q}, 0; t) = -f_y^*(t, \bar{q}, \bar{y})$, then,

$$
\begin{aligned}
I_1(\phi) &= \int_0^T \frac{a(t, q_\lambda; y_\lambda, \phi) - a(t, \bar{q}; y_\lambda, \phi)}{\lambda}\, dt \\
&\le \int_0^T \left\langle \frac{(A(t, q_\lambda) - A(t, \bar{q}))\phi}{\lambda}, \bar{y} \right\rangle_{V',V} dt \\
&\quad + \int_0^T \left\langle \frac{(A(t, q_\lambda) - A(t, \bar{q}))\phi}{\lambda}, y_\lambda - \bar{y} \right\rangle_{V',V} dt \\
&= J_1 + J_2.
\end{aligned}
$$

By (C2), we have

$$
\lim_{\lambda \to 0} J_1 = \int_0^T a'_q(t, \bar{q}; \bar{y}, \phi)(q - \bar{q})\, dt.
$$

Also by (C2), for some $\theta \in (0,1)$, we have

$$
J_2 = \int_0^T a'_q(t, \bar{q} + \theta\lambda(q - \bar{q}); y_\lambda - \bar{y}, \phi)(q - \bar{q})\, dt.
$$

By the strong convergence of $y_\lambda \to \bar{y}$ in (3.33), we have

$$J_2 \leq \tilde{\gamma} \|\phi\|_{L^2(0,T;H)} \|y_\lambda - \bar{y}\|_{L^2(0,T;H)} \|q - \bar{q}\|_Q \to 0, \qquad \text{as } \lambda \to 0.$$

Hence,

$$\lim_{\lambda \to 0} I_1 = \int_0^T a_q' (t, \bar{q}; \bar{y}, \phi) (q - \bar{q}) \, dt. \tag{3.42}$$

By similar calculation as in the above, it follows from (C1) that

$$\begin{aligned} \lim_{\lambda \to 0} I_2 (\phi) &= \lim_{\lambda \to 0} \int_0^T \left(\frac{f (t, q_\lambda; y (q)) - f (t, \bar{q}; y (\bar{q}))}{\lambda}, \phi \right) dt \\ &= \int_0^T \left(f_q' (t, \bar{q}, \bar{y}) (q - \bar{q}), \phi \right) dt. \end{aligned} \tag{3.43}$$

Finally, for fixed a.e., $t \in [0, T]$, by (C1) we have, for all $\theta \in [0, 1]$, that,

$$\lim_{\lambda \to 0} f_y^* (t, q_\lambda; \theta y_\lambda + (1 - \theta) \bar{y}) = f_y^* (t, \bar{q}; \bar{y})$$

in $\mathcal{L}(H)$. Since the convergence is uniform on $[0, 1]$, we also have

$$\lim_{\lambda \to 0} L (q, \lambda; t) = -f_y^* (t, \bar{q}, \bar{y}), \qquad \text{a.e., } t \in [0, 1].$$

Applying the Lebesgue dominated convergence theorem, we have

$$\lim_{\lambda \to 0} \int_0^T (z_\lambda, L (q, \lambda; t) \phi) \, dt = - \int_0^T \left(z, f_y^* (t, \bar{q}, \bar{y}) \phi \right) dt. \tag{3.44}$$

Therefore, taking $\lambda \to 0$ in (3.39), and using (3.40)–(3.44), one sees that z satisfies

$$\begin{gathered} (z (T), \phi (T))_H + \int_0^T \langle z, -\phi' + A^* (t, \bar{q}) \phi - f_y^* (t, \bar{q}; y (\bar{q})) \phi \rangle_{V,V'} \, dt \\ = - \int_0^T a_q' (t, \bar{q}; y (\bar{q}), \phi) (q - \bar{q}) \, dt + \int_0^T \left(f_q' (t, \bar{q}; y (\bar{q})) (q - \bar{q}), \phi \right) dt, \\ z (0) = 0, \end{gathered} \tag{3.45}$$

for all $\phi \in X[\bar{q}, 0] \subset W(0, T)$. Therefore, we can define l on $X[\bar{q}, 0]$ with $L(\bar{q}, 0; t] = -f_y^*(t, \bar{q}, \bar{y})$ as

$$l (\phi) = - \int_0^T a_q' (t, \bar{q}; y (\bar{q}), \phi) (q - \bar{q}) \, dt + \int_0^T \left(f_q' (t, \bar{q}; y (\bar{q})) (q - \bar{q}), \phi \right) dt.$$

This means that l is a bounded linear functional on $X[\bar{q}, 0] \subset W(0, T)$, satisfying

$$(z (T), \phi (T))_H + \int_0^T \langle z, \Phi [\bar{q}, 0; t] (\phi) \rangle_{V,V'} \, dt = l (\phi),$$

where z is the unique solution of (3.45). This proves Theorem 4.

REMARK 2. From the above proof, we see that $z = y'(\bar{q}, q - \bar{q})$ is linear for the variable $q - \bar{q}$. Therefore, $y'(\bar{q}, q - \bar{q}) = y'(\bar{q})(q - \bar{q})$.

Next, we look for a necessary condition for optimal parameter $\bar{q}$. Calculating the Gâteaux derivative of the cost (3.22), the optimality condition (3.24) of optimal parameter $\bar{q}$ is rewritten as

$$J' (\bar{q}) (q - \bar{q}) \geq 0, \qquad \forall q \in Q_{\text{ad}}. \tag{3.46}$$

By Theorem 4, $y(q)$ is weakly Gâteaux differentiable at $\bar{q}$ in the direction $q-\bar{q}$ and (3.46) can be rewritten as

$$\langle \mathcal{C}^* \Lambda_{\mathcal{M}} \left(\mathcal{C} y\left(\bar{q}\right) - z_d\right), y'\left(\bar{q}\right)\left(q-\bar{q}\right)\rangle_{W(0,T)',W(0,T)} \geq 0, \qquad \forall\, q \in Q_{\mathrm{ad}},$$

where z_d is the desired value of $y(\bar{q})$ in observation space $\mathcal{M}$, $\Lambda_{\mathcal{M}}$ is the canonical isomorphism from $\mathcal{M}$ to $\mathcal{M}'.y'(\bar{q})(q-\bar{q})$ is weakly Gâteaux differential of $y(q)$ at $\bar{q}$ in the direction $q-\bar{q}$.

To avoid the complexity of observation state, we consider distributed observations and terminal value observations as in Lions [17]. That is, consider the following two cases.

1. Take $\mathcal{C} \in \mathcal{L}(L^2(0,T;V),\mathcal{M})$ and observe $z(q) = \mathcal{C}y(q)$.
2. Take $\mathcal{C} \in \mathcal{L}(H;\mathcal{M})$ and observe $z(q) = \mathcal{C}y(T;q)$.

For each case, we introduce an adjoint state system, and form the condition (3.47), we derive necessary condition of optimality, which solves the problem (ii) in a satisfactory manner.

1. Case of $\mathcal{C} \in \mathcal{L}(L^2(0,T;V),\mathcal{M})$. In this case, the cost function is given as

$$J\left(q\right) = \left\|\mathcal{C}y\left(q\right) - z_d\right\|_{\mathcal{M}}^2, \qquad \forall\, q \in Q. \tag{3.48}$$

Then, it is easily verified that the optimality condition (3.47) for optimal parameter $\bar{q}$ is

$$\left(\mathcal{C}y\left(\bar{q}\right) - z_d, \mathcal{C}z\right)_{\mathcal{M}} \geq 0, \qquad \forall\, q \in Q_{\mathrm{ad}}, \tag{3.49}$$

where $z = y'(\bar{q})(q-\bar{q})$, $\bar{q}$ is the optimal parameter for (3.48). Using the isomorphism $\Lambda_{\mathcal{M}}$, we can transfer the condition (3.49) to

$$\int_0^T \langle \mathcal{C}^* \Lambda_{\mathcal{M}} \left(\mathcal{C}y\left(\bar{q}\right) - z_d\right), z\rangle_{V',V}\, dt \geq 0, \qquad \forall\, q \in Q_{\mathrm{ad}}. \tag{3.50}$$

We introduce the adjoint system by

$$\begin{aligned} -\frac{d}{dt} p\left(\bar{q}\right) + A^*\left(t,\bar{q}\right) p\left(\bar{q}\right) &= f_y^*\left(t, y\left(\bar{q}\right)\right) p\left(\bar{q}\right) + \mathcal{C}^* \Lambda_{\mathcal{M}} \left(\mathcal{C}y\left(\bar{q}\right) - z_d\right), \qquad \text{in } (0,T), \\ p\left(T,\bar{q}\right) &= 0, \end{aligned} \tag{3.51}$$

where $p(\bar{q})$ denotes an adjoint state depending on optimal parameter $\bar{q}$. Since $f_y^*(t,y(\bar{q}))p(\bar{q}) \in L^2(0,T;H)$ and $\mathcal{C}^*\Lambda_{\mathcal{M}}(\mathcal{C}y(\bar{q}) - z_d) \in L^2(0,T;H)$ as $y(\bar{q}) \in H$ in the two terms, then (3.51) is a linear equation of $p(\bar{q})$. Then by Theorem 1, there is a unique weak solution $p(\bar{q}) \in W(0,T)$.

Theorem 5. *Let $\mathcal{C} \in \mathcal{L}(L^2(0,T;V),\mathcal{M})$. Assume that all the conditions of Theorem 4 hold. Then, the optimal parameter $\bar{q} \in Q_{\mathrm{ad}}$ for (3.48) is characterized by*

$$\begin{aligned} \frac{dy\left(\bar{q}\right)}{dt} + A\left(t,\bar{q}\right) y\left(\bar{q}\right) &= f\left(t, y\left(\bar{q}\right)\right), \qquad \text{in } (0,T), \\ y\left(0,\bar{q}\right) &= y_0 \in H; \end{aligned}$$

$$\begin{aligned} -\frac{dp\left(\bar{q}\right)}{dt} + A^*\left(t,\bar{q}\right) p\left(\bar{q}\right) &= f_y^*\left(t, y\left(\bar{q}\right)\right) p\left(\bar{q}\right) + \mathcal{C}^* \Lambda_{\mathcal{M}} \left(\mathcal{C}y\left(\bar{q}\right) - z_d\right), \qquad \text{in } (0,T), \\ p\left(T,\bar{q}\right) &= 0 \in H; \end{aligned}$$

$$-\int_0^T a_q'\left(t,\bar{q}; y\left(\bar{q}\right), p\left(\bar{q}\right)\right)\left(q-\bar{q}\right) dt + \int_0^T \left(f_q'\left(t,\bar{q},y\left(\bar{q}\right)\right), \left(q-\bar{q}\right), p\left(\bar{q}\right)\right) dt \geq 0, \qquad \forall\, q \in Q_{\mathrm{ad}}$$

with

$$y\left(\bar{q}\right) \in W\left(0,T\right), \qquad p\left(\bar{q}\right) \in W\left(0,T\right).$$

2. CASE OF $\mathcal{C} \in \mathcal{L}(H, \mathcal{M})$. In this case, the cost function is expressed as

$$J(q) = \left\| \mathcal{C}y(T,q) - z_d^T \right\|_{\mathcal{M}}^2, \qquad \forall q \in Q, \tag{3.52}$$

where $z_d^T \in \mathcal{M}$ is a desired value. Then, the optimal parameter $\bar{q}$ for (3.52) is characterized by

$$\langle \mathcal{C}^* \Lambda_{\mathcal{M}} \left(\mathcal{C}y(T,\bar{q}) - z_d^T \right), z(T) \rangle_{V',V} \geq 0, \qquad \forall q \in Q_{\text{ad}}, \tag{3.53}$$

where $z(T) = y'(T,\bar{q})(q - \bar{q})$. For the terminal value observation cost (3.52), we introduce an adjoint system defined by

$$\begin{aligned} -\frac{d}{dt} p(\bar{q}) + A^*(t,\bar{q}) p(\bar{q}) = f_y^*(t, y(\bar{q})) p(\bar{q}), \qquad \text{in } (0,T), \\ p(T,\bar{q}) = \mathcal{C}^* \Lambda_{\mathcal{M}} \left(\mathcal{C}y(T,\bar{q}) - z_d^T \right) \in H. \end{aligned} \tag{3.54}$$

Since $\mathcal{C}^* \Lambda_{\mathcal{M}}(\mathcal{C}y(T,\bar{q}) - z_d) \in H$, then (3.54) is a well-posed linear equation of $p(\bar{q})$ and permits a unique weak solution $p(\bar{q})$ in $W(0,T)$, if the change of time variable $t \to T - t$ is adapted.

THEOREM 6 LET $\mathcal{C} \in \mathcal{L}(H, \mathcal{M})$. Assume that all the conditions in Theorem 4 hold. Then, the optimal parameter $\bar{q}$ for (3.52) satisfies

$$\begin{aligned} \frac{d}{dt} y(\bar{q}) + A(t,\bar{q}) y(\bar{q}) = f(t, y(t,\bar{q})), \qquad \text{in } (0,T), \\ y(0,\bar{q}) = y_0 \in H; \end{aligned}$$

$$\begin{aligned} -\frac{d}{dt} p(\bar{q}) + A^*(t,\bar{q}) p(\bar{q}) = f_y^*(t, y(t,\bar{q})) p(\bar{q}), \qquad \text{in } (0,T), \\ p(T,\bar{q}) = \mathcal{C}^* \Lambda_{\mathcal{M}} \left(\mathcal{C}y(T,\bar{q}) - z_d^T \right) \in H; \end{aligned}$$

$$-\int_0^T a_q'(t,\bar{q}; y(\bar{q}), p(\bar{q}))(q - \bar{q})\, dt + \int_0^T \left(f_q'(t,\bar{q}, y(\bar{q}))(q - \bar{q}), p(\bar{q}) \right) dt \geq 0, \qquad \forall q \in Q_{\text{ad}}$$

with

$$y(\bar{q}), p(\bar{q}) \in W(0,T).$$

REMARK 3. It is better to notice that Theorems 5 and Theorem 6 can not be applied to concrete nonlinear parabolic equation in which H is taken to be $L^2(\Omega)$. In fact, a well known result due to Krasnoselskii states that the mapping $y \mapsto f(\cdot, y)$ is Fréchet differentiable in $H = L^2(\Omega)$ if only f is affine-linear. Then assumption (C1) makes no sense in these cases.

4. CONCLUSIONS

This paper studied the parameter identification problem of nonlinear parabolic distributed parameter system via the variational method. For the output error criterion, given by the quadratic cost, the existence of optimal parameter is proved. Finally, using the transposition method, the necessary condition for the optimal parameter is given for the case of distributed observation and terminal observation.

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