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Numerical solution of damped nonlinear Klein–Gordon equations using variational method and finite element approach

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Abstract

Numerical treatment for damped nonlinear Klein–Gordon equations, based on variational method and finite element approach, is studied. A semi-discrete algorithm is proposed by using quadratic interpolation functions of continuous time and spatial dimension one. The Gauss–Legendre quadrature has been utilized for numerical integrations of nonlinear terms, and Runge–Kutta method is used for solving ordinary differential equation. Finally, three dimensional graphics of numerical solutions are used to demonstrate the numerical results.
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Keywords: Klein–Gordon equations; Numerical solution; Finite element methods; Gauss–Legendre quadrature; Runge–Kutta method

1. Introduction

This paper deals with the numerical solution of the damped nonlinear Klein–Gordon equations with Dirichlet boundary condition using variational method and finite element approximation. The equation is one of the nonlinear wave equation arising in relativistic quantum mechanics. The Klein–Gordon

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equation with damping term is described by the following hyperbolic partial differential equation with second order derivative in time

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \delta |y|^\gamma y = f, \tag{1}$$

where $\alpha, \beta, \gamma > 0, \delta \in \mathbf{R}$ are physical constants and f is the time varying external input.

The Klein–Gordon equation has been studied in many literatures, e.g., [8,11,15,20]. However, numerical treatment for Klein–Gordon equation is rarely reported (cf. [4,5,9,12,17]). Particularly, [9] concern with the decomposition method and difference method used in [12] nonlinear problems. The purpose of this paper is to investigate its numerical solution for damped nonlinear problems of dimension one. Comparing with other nonlinear hyperbolic partial differential equations, the difficulty in solving Eq. (1) arises from the unboundedness of the nonlinear input $|y|^\gamma y$. The new method proposed in this paper can overcome this obstacle.

The contents of this paper is as follows. Section 2 introduces the notations and mathematical setting for the problems. In Section 3, we state the existence of approximate solution for Klein–Gordon equation using the variational formulation (cf. [3]). In Section 4, we construct a numerical solution based on finite element method using quadratic interpolation functions. In Section 5, we give the convergence theorem for this solution. Lastly, in Section 6, we do some numerical simulations and present some 3D graphics of the numerical solution for various parameters.

2. Notations and mathematical setting

Let $\Omega = (0, l)$ be an open bounded set in R^1 and $Q = (0, T) \times (0, l)$. We consider the damped Klein–Gordon equation described by

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \delta g(y) = f \text{ in } Q, \tag{2}$$

where $\alpha, \beta, \delta \in \mathbf{R}$ are constants representing the gratitude of damping, diffusion and nonlinearity effects, Δ is a Laplacian, $g(y) = |y|^\gamma y$, $\gamma > 0$ is a nonlinear function and f is the force. We pose the Dirichlet condition

$$y(t, 0) = y(t, l) = 0, \text{ on } [0, T] \tag{3}$$

and the initial values are given by

$$y(0, x) = y_0(x) \text{ in } (0, l), \qquad \frac{\partial y}{\partial t}(0, x) = y_1(x) \text{ in } (0, l). \tag{4}$$

Define two Hilbert spaces $H = L^2(0,l)$ and $V = H_0^1(0,l)$, according to the Dirichlet boundary condition (3). H and V are endowed with the usual inner products and norms as

$$(\psi,\phi) = \int_0^l \psi(x)\phi(x)\,\mathrm{d}x, \quad |\psi| = (\psi,\psi)^{1/2}, \quad \text{for all } \phi,\psi \in L^2(0,l),$$

$$((\psi,\phi)) = \int_0^l \frac{\partial\psi(x)}{\partial x}\frac{\partial\phi(x)}{\partial x}\,\mathrm{d}x, \quad \|\psi\| = ((\psi,\psi))^{1/2}, \quad \text{for all } \phi,\psi \in H_0^1(0,l).$$

Then the pair (V,H) is a Gelfand triple space with a notation, $V \hookrightarrow H \hookrightarrow V'$, which means that embeddings $V \subset H$ and $H \subset V'$ are continuous, dense and compact. To use a variational formulation, let us introduce the bilinear form

$$a(\phi,\varphi) = \int_0^l \nabla\phi\cdot\nabla\varphi\,\mathrm{d}x = ((\phi,\varphi)), \quad \forall\phi,\varphi \in H_0^1(0,l).$$

Here a is symmetric, bounded on $H_0^1(0,l)\times H_0^1(0,l)$, satisfying $a(\phi,\phi) \geqslant \|\phi\|^2$ for $\forall\phi \in H_0^1(0,l)$. Then we can define a bounded operator $A \in \mathscr{L}(V,V')$ by the relation $a(\phi,\varphi) = \langle A\phi,\varphi\rangle$. This operator A is an isomorphism from V onto V', and the restriction of A on H is self adjoint and has a dense domain $\mathscr{D}(A) = \{\phi \in V | A\phi \in H\}$ (cf. [18]).

Lemma 1. *If $y \in H_0^1(0,l)$, then $y \in L^{2\gamma+2}(0,l)$, that is $g(y) \in L^2(0,l)$ for arbitrary $\gamma > 0$, where $g(y) = |y|^\gamma y$.*

Proof. We recall Gagliardo–Nirenberg inequality (cf. [20]), for $\forall y \in H_0^1(0,l)$ and arbitrary $p > 1$, there exists $C > 0$ such that

$$\left(\int_0^l |y|^{\frac{2p}{p-1}}\,\mathrm{d}x\right)^{2(p-1)} \leqslant C\left(\int_0^l |y|^2\,\mathrm{d}x\right)^{2p-1}\left(\int_0^l |\nabla y|\,\mathrm{d}x\right). \tag{5}$$

Taking $p = 1 + \frac{1}{\gamma} > 1$ such that $\frac{2p}{p-1} = 2\gamma + 2$, then by (5) we have that

$$\|y\|_{L^{2\gamma+2}(0,l)}^{\frac{4\gamma+4}{\gamma}} \leqslant C\|y\|_H^{\frac{2\gamma+4}{\gamma}}\|\nabla y\|_H. \tag{6}$$

Then (6) implies that

$$\|y\|_{L^{2\gamma+2}(0,l)} \leqslant \|y\|_H^{\frac{2\gamma+4}{4\gamma+4}}\|y\|_{H_0^1(0,l)}^{\frac{2\gamma}{4\gamma+4}} \leqslant \|y\|_V.$$

It means that

$$g(y) \in L^2(0,l), \quad \text{for } \forall y \in V. \tag{7}$$

This completes the proof. □

Remark 2. Lemma 1 means that the exponent γ, $0 \leqslant \gamma < \infty$ can be taken arbitrary when $n = 1$. We infer that the nonlinear operator $g : H_0^1(0, l) \to L^2(0, l), \phi \to g(\phi)$ is well defined.

The problem (2)–(4) is reduced to the following Cauchy problem in H:

$$\begin{cases} \dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \alpha \dfrac{\mathrm{d}y}{\mathrm{d}t} - \beta \Delta y + \delta g(y) = f \quad \text{in } (0, T), \\ y(0) = y_0 \in V, \quad \dfrac{\mathrm{d}y}{\mathrm{d}t}(0) = y_1 \in H. \end{cases} \tag{8}$$

For general treatment of the nonlinear second order equations of hyperbolic type, we refer to [13,14,20].

3. Existence of weak approximate solution

In this section, we consider the existence of weak approximate solution in the framework of variational methods (cf. [3,7]). For simplicity, we shall write $g' = \dfrac{\mathrm{d}g}{\mathrm{d}t}$, $g'' = \dfrac{\mathrm{d}^2 g}{\mathrm{d}t^2}$ and define the solution space by

$$W(0, T) = \{g | g \in L^2(0, T; V), g' \in L^2(0, T; H), g'' \in L^2(0, T; V')\}.$$

Now we give the definition of weak solutions of the problem (cf. [3,20]).

Definition 3. A function y is said to be a weak solution of (8) if $y \in W(0, T)$ satisfies

$$\begin{cases} \langle y''(\cdot), \phi \rangle_{V',V} + \alpha(y'(\cdot), \phi) + \beta((y(\cdot), \phi)) + \delta(|y(\cdot)|^\gamma y(\cdot), \phi) = (f(\cdot), \phi) \\ \quad \text{for all } \phi \in V \quad \text{in the sense of } \mathscr{D}'(0, T), \\ y(0) = y_0, \quad \dfrac{\mathrm{d}y}{\mathrm{d}t}(0) = y_1. \end{cases}$$

Here the symbol $\langle \cdot, \cdot \rangle_{V',V}$ denotes a dual pairing between V and V'. $\mathscr{D}'(0, T)$ denotes the space of distributions on $(0, T)$.

Remark 4. By (7) in Lemma 1, we note that $|y(t)|^\gamma y(t) \in H$ a.e. $t \in [0, T]$ if $y \in W(0, T)$. The nonlinear term is meaningful in weak form (9).

Next we construct an approximate solution for the system (8). Since the embedding of V into H is compact, then there exists an orthogonal basis of H, $\{w_j\}_{j=1}^\infty$ consisting of eigenfunctions of $A = \Delta$, such that

$$\begin{cases} A w_j = \lambda_j w_j, \quad \forall j, \\ 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty. \end{cases} \tag{9}$$

We denote by P_m the orthogonal projection of H(or V) onto the space spanned by $\{w_1,\dots,w_m\}$. We implement a Faedo–Galerkin method as used in [3]. For each $m\in N$, we define an approximate solution of the problem (8) by

$$y_m(t)=\sum_{j=1}^{m} g_{jm}(t)w_j. \tag{10}$$

Then the approximate solution $y_m(t)$ satisfies the approximate equation given by

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(y_m(t),w_j)+\alpha\dfrac{\mathrm{d}}{\mathrm{d}t}(y_m(t),w_j)+\beta((y_m(t),w_j)) \\ \quad +\delta(|y_m(t)|^{\gamma}y_m(t),w_j)=(f(t),w_j), \quad t\in[0,T], \\ y_m(0)=P_my_0, \quad \dfrac{\mathrm{d}}{\mathrm{d}t}y_m(0)=P_my_1, \quad 1\leqslant j\leqslant m. \end{cases} \tag{11}$$

We set $y_{0m}=P_my_0$ and $y_{1m}=P_my_1$. Then

$$y_{0m}\to y_0 \quad \text{in } V, \quad y_{1m}\to y_1 \quad \text{in } H \quad \text{as } m\to\infty.$$

Based on Lemma 1, we show the local Lipschitz continuity for nonlinear term (cf. [15,20]).

Lemma 5. *The operator g is locally Lipschitz from V into H. That is, there exists a constant $k>0$ such that*

$$|g(\psi)-g(\varphi)|\leqslant k(\|\psi\|+\|\varphi\|)^{\gamma}\|\psi-\varphi\|, \quad \forall\psi,\varphi\in V. \tag{12}$$

We state the following local existence theorem of the weak solutions (cf. [15]).

Theorem 6. *Let $f\in L^2(0,T;H)$, $y_0\in V$, $y_1\in H$. Then the problem* (8) *with $\alpha,\beta>0$, $\delta\in\mathbf{R}$ and γ arbitrary, has a unique weak approximate solution y_m in $W(0,T)$.*

Proof. For each $m\in N$, we define an approximate solution $y_m(t)$ of the problem (8) by (10), it also satisfies (11). Therefore, Eq. (11) can be written as m vector differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{g}_m+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\vec{g}_m+\beta\Lambda\vec{g}_m=\vec{k}(t,\vec{g}_m) \tag{13}$$

with initial values

$$\vec{g}_m(0)=\begin{bmatrix}(y_{0m},w_1)\\(y_{0m},w_2)\\ \vdots \\ (y_{0m},w_m)\end{bmatrix}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\vec{g}_m(0)=\begin{bmatrix}(y_{1m},w_1)\\(y_{1m},w_2)\\ \vdots \\ (y_{1m},w_m).\end{bmatrix}.$$

Here $\vec{g}_m = [g_{1m}, \dots, g_{mm}]^{\mathrm{t}}$, $\Lambda = \mathrm{diag}(\lambda_i : i = 1, \dots, m)$, and

$$\vec{k}(t, \vec{g}_m) = \begin{bmatrix} (f(t), w_1) - \delta\left(\left|\sum_{j=1}^{m} g_{jm} w_j\right|^{\gamma} \sum_{j=1}^{m} g_{jm} w_j, w_1\right) \\ (f(t), w_2) - \delta\left(\left|\sum_{j=1}^{m} g_{jm} w_j\right|^{\gamma} \sum_{j=1}^{m} g_{jm} w_j, w_2\right) \\ \vdots \\ (f(t), w_m) - \delta\left(\left|\sum_{j=1}^{m} g_{jm} w_j\right|^{\gamma} \sum_{j=1}^{m} g_{jm} w_j, w_m\right) \end{bmatrix},$$

where $[\cdots]^{\mathrm{t}}$ denotes the transpose of $[\cdots]$. The nonlinear forcing function vector $\vec{k}$ is locally Lipschitz continuous. Indeed, for $\vec{g}_m = [g_{1m}, \dots, g_{mm}]^{\mathrm{t}}, \vec{h}_m = [h_{1m}, \dots, h_{mm}]^{\mathrm{t}}$ it follows from inequality (12) in Lemma 5 and Schwartz inequality that

$$\begin{aligned} \left|\vec{k}(t, \vec{g}_m) - \vec{k}(t, \vec{h}_m)\right|^2 &= \delta^2 \sum_{i=1}^{m} \left|\left(g\left(\sum_{j=1}^{m} g_{jm} w_j\right) - g\left(\sum_{j=1}^{m} h_{jm} w_j\right), w_i\right)\right|^2 \\ &\leqslant \delta^2 m \left|(g\left(\sum_{j=1}^{m} g_{jm} w_j\right) - g\left(\sum_{j=1}^{m} h_{jm} w_j\right)\right|^2 \\ &\leqslant \delta^2 k^2 m \left(\left\|\sum_{j=1}^{m} g_{jm} w_j\right\| + \left\|\sum_{j=1}^{m} h_{jm} w_j\right\|\right)^{\gamma} \left\|\sum_{j=1}^{m} (g_{jm} - h_{jm}) w_j\right\| \\ &\leqslant \delta^2 k^2 m \left(\sum_{j=1}^{m} \lambda_j\right)^{\frac{\gamma+1}{2}} (|\vec{g}_m| + |\vec{h}_m|)^{\gamma} \cdot |\vec{g}_m - \vec{h}_m|. \end{aligned}$$

Therefore, by reducing (13) to a first order system and applying Carathéodory type existence theorem, there exists a $T > 0$ such that this second order vector differential equation (13) admits a local unique solution $\vec{g}_m$ on $[0, T]$. Hence we can construct the approximate solutions $y_m(t)$ of (11) in the form (10) on $[0, T]$. This completes the proof. □

We state global existence of weak solution (cf. [15,20]).

Theorem 7. *Let* $f \in L^2(0, T; H)$, $y_0 \in V$, $y_1 \in H$. *Then the problem* (8) *with* $\alpha, \beta > 0$, $\gamma > 0$ *and* $\delta \geqslant 0$ *arbitrary, has a unique global weak solution* y_m *in* $W(0, T)$.

4. Finite element approach

Let $Q = [0, T] \times (0, l)$. We construct numerical solution to the following one dimensional Klein–Gordon equation using finite element method (cf. [2,6,7,21]). We recall the system described by

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + \alpha \dfrac{\partial y}{\partial t} - \beta \Delta y + \delta |y|^\gamma y = f & \text{in } [0,T] \times (0,l), \\ y(t,0) = y(t,l) = 0 & \text{on } [0,T], \\ y(0,x) = y_0(x), \quad y'(0,x) = y_1(x) & \text{in } (0,l), \end{cases} \tag{14}$$

where $\alpha, \beta > 0$, $\delta \in \mathbf{R}$ and $\gamma > 0$ are constants.

Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = l$ be a partition of the interval $[0,l]$ into subintervals $I_e = [x_{e-1}, x_e]$ of length $h^e = x_e - x_{e-1}$, $e = 1,2,\dots,N+1$. Let V_h be a set of functions ϕ_i^e, $i = 1,2,3$. $e = 1,2,\dots,N+1$ such that ϕ_i^e is quadratic function on each interval I_e, $e = 1,2,\dots,N+1$, and continuous on $[0,l]$ with $\psi_i^e(0) = \psi_i^e(l) = 0$. Then it is clear that $V_h \subset H_0^1(0,l)$. Let us introduce a set of quadratic interpolation functions (cf. [22]) $\psi_i^e \in V_h$ as

$$\begin{cases} \psi_1^e(x) = \left(1 - \dfrac{x - x_e}{h^e}\right)\left(1 - \dfrac{2(x - x_e)}{h^e}\right), \\ \psi_2^e(x) = \dfrac{4(x - x_e)}{h^e}\left(1 - \dfrac{x - x_e}{h^e}\right), \\ \psi_3^e(x) = -\dfrac{(x - x_e)}{h^e}\left(1 - \dfrac{2(x - x_e)}{h^e}\right), \quad e = 1,2,\dots,N. \end{cases}$$

Assume $N = 11$, $e = 6$ and $l = 1$, the figure of ψ_i^e for $i = 1,2,3$ on $[0,1]$ is shown in Fig. 1.

The interpolation functions satisfy the following properties, which are known as the interpolation properties:

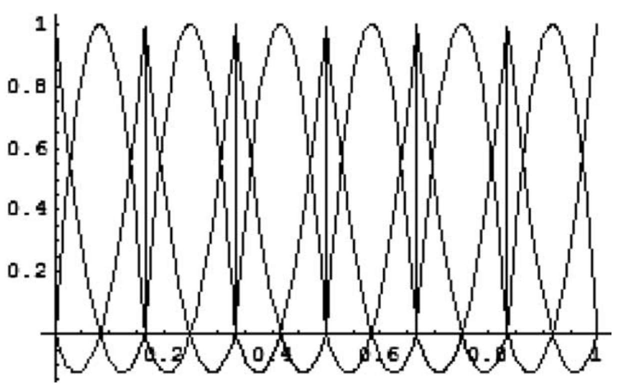


Fig. 1. Quadratic interpolation functions.

$$\text{(a) } \psi_i^e(x_{e-1}) = \begin{cases} 0 & \text{if } i \neq 1; \\ 1 & \text{if } i = 1, \end{cases} \quad \psi_i^e(x_e) = \begin{cases} 0 & \text{if } i \neq 3; \\ 1 & \text{if } i = 3, \end{cases} \tag{15}$$

$$\text{(b) } \sum_{i=1}^{3} \psi_i^e(x) = 1, \quad \sum_{i=1}^{3} \frac{d\psi_i^e}{dx} = 0.$$

The e-th element of approximate solution is defined by

$$y_h^e(t,x) = \sum_{i=1}^{3} \xi_i^e(t)\psi_i^e(x), \quad e = 1, 2, \ldots, N.$$

Here $\xi_i^e(t)$ is continuous with respect to t on $[0, T]$. According to the Dirichlet boundary, we set $\xi_1^1(t) = \xi_3^N(t) = 0$. Then the total approximate solution can be represented as

$$y_h(t,x) = \sum_{e=1}^{N} y_h^e(t,x) = \sum_{e=1}^{N} \sum_{i=1}^{3} \xi_i^e(t)\psi_i^e(x) \in V_h, \quad \forall t \in [0, T].$$

Thus by (14), y_h^e satisfies

$$\begin{cases} (y_h^{e''}, \psi_j^e) + \alpha(y_h^{e'}, \psi_j^e) + \beta(\nabla y_h^e, \nabla\psi_i^e) + \delta(|y_h^e|^\gamma y_h^e, \psi_j^e) = (f, \psi_j^e), \\ (y_h^e(0), \psi_j^e) = (y_0, \psi_j^e), \quad (y_h^{e'}(0), \psi_j^e) = (y_1, \psi_j^e), \quad e = 1, \ldots, N. \end{cases}$$

Here $\xi_i^e(t)$ satisfies the following second order differential equations:

$$\begin{cases} \sum\limits_{i=1}^{3} \xi_i^{e''}(t)(\psi_i^e, \psi_j^e) + \alpha \sum\limits_{i=1}^{3} \xi_i^{e'}(t)(\psi_i^e, \psi_j^e) \\ \quad + \beta \sum\limits_{i=1}^{3} \xi_i^e(t)(\nabla\psi_i^e, \nabla\psi_j^e) + \delta\left(\sum\limits_{i=1}^{3} |\xi_i^e \psi_i^e|^\gamma \xi_i^e \psi_i^e, \psi_j^e\right) = (f, \psi_j^e), \\ \sum\limits_{i=1}^{3} \xi_i^e(0)(\psi_i^e, \psi_j^e) = (y_0, \psi_j^e), \sum\limits_{i=1}^{3} \xi_i^{e'}(0)(\psi_i, \psi_j^e) = (y_1, \psi_j^e), \quad e = 1, \ldots, N. \end{cases} \tag{16}$$

Set

$$\begin{aligned}
&\Psi^e = (\psi_i^e, \psi_j^e)_{i=1,2,3}^{j=1,2,3} \in M_{3\times 3}(\mathbf{R}),\\
&\Phi^e = (\nabla\psi_i^e, \nabla\psi_j^e)_{i=1,2,3}^{j=1,2,3} \in M_{3\times 3}(\mathbf{R}),\\
&\Xi^e(t) = [\xi_i^e(t), \xi_2^e(t), \xi_3^e(t)]^{\mathrm{t}} \in M_{3\times 1}(\mathbf{R}),\\
&\Xi^{e'}(t) = [\xi_i^{e'}(t), \xi_2^{e'}(t), \xi_3^{e'}(t)]^{\mathrm{t}} \in M_{3\times 1}(\mathbf{R}),\\
&\Xi^{e''}(t) = [\xi_i^{e''}(t), \xi_2^{e''}(t), \xi_3^{e''}(t)]^{\mathrm{t}} \in M_{3\times 1}(\mathbf{R}),\\
&F^e(t) = [(f(t), \psi_1^e), (f(t), \psi_2^e), (f(t), \psi_3^e)]^{\mathrm{t}} \in M_{3\times 1}(\mathbf{R}),\\
&Y_0^e = [(y_0, \psi_1^e), (y_0, \psi_2^e), (y_0, \psi_3^e)]^{\mathrm{t}} \in M_{3\times 1}(\mathbf{R}),\\
&Y_1^e = [(y_1, \psi_1^e), (y_1, \psi_2^e), (y_1, \psi_3^e)]^{\mathrm{t}} \in M_{3\times 1}(\mathbf{R})
\end{aligned}$$

and

$$G^e(\Xi) = \begin{bmatrix} (|\xi_1^e\psi_1^e|^\gamma \xi_1^e\psi_1^e, \psi_1^e) \\ (|\xi_2^e\psi_2^e|^\gamma \xi_2^e\psi_2^e, \psi_2^e) \\ (|\xi_3^e\psi_3^e|^\gamma \xi_3^e\psi_3^e, \psi_3^e) \end{bmatrix} \in M_{3\times 3}(\mathbf{R}). \tag{17}$$

Then Eq. (16) can be expressed in the vector form as

$$\Psi^e\Xi^{e''} + \alpha\Psi^e\Xi^{e'} + \beta\Phi^e\Xi^e + \delta G^e = F^e. \tag{18}$$

By the continuity of $\xi_i^e(t)$ on $[0, T]$, we have $\xi_3^e(t) = \xi_1^{e+1}(t)$ for $e = 1, 2, \dots, N$. To assemble Eq. (18) into an overall equation, we introduce the following matrixes and vectors:

$$\Psi = \begin{bmatrix}
\psi_{11}^1 & \psi_{12}^1 & \psi_{13}^1 & & & & & \\
\psi_{21}^1 & \psi_{22}^1 & \psi_{23}^1 & & & & & \\
\psi_{31}^1 & \psi_{32}^1 & \psi_{33}^1 + \psi_{11}^2 & \psi_{12}^2 & \psi_{13}^2 & & 0 & \\
 & & \psi_{21}^2 & \psi_{22}^2 & \psi_{23}^2 & & & \\
 & & \psi_{31}^2 & \psi_{32}^2 & \psi_{33}^2 + \psi_{11}^3 & & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & 0 & & & \psi_{33}^{N-1} + \psi_{11}^N & \psi_{12}^N & \psi_{13}^N \\
 & & & & & \psi_{21}^N & \psi_{22}^N & \psi_{23}^N \\
 & & & & & \psi_{31}^N & \psi_{32}^N & \psi_{33}^N
\end{bmatrix},$$

$$\Phi = \begin{bmatrix} \phi_{11}^1 & \phi_{12}^1 & \phi_{13}^1 & & & & & \\ \phi_{21}^1 & \phi_{22}^1 & \phi_{23}^1 & & & & & \\ \phi_{31}^1 & \phi_{32}^1 & \phi_{33}^1+\phi_{11}^2 & \phi_{12}^2 & \phi_{13}^2 & & 0 & \\ & & \phi_{21}^2 & \phi_{22}^2 & \phi_{23}^2 & & & \\ & & \phi_{31}^2 & \phi_{32}^2 & \phi_{33}^2+\phi_{11}^3 & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & 0 & & & \phi_{33}^{N-1}+\phi_{11}^N & \phi_{12}^N & \phi_{13}^N \\ & & & & & \phi_{21}^N & \phi_{22}^N & \phi_{23}^N \\ & & & & & \phi_{31}^N & \phi_{32}^N & \phi_{33}^N, \end{bmatrix},$$

$$\Xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \vdots \\ \xi_{2N-1} \\ \xi_{2N} \\ \xi_{2N+1} \end{bmatrix} = \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_3^1(=\xi_1^2) \\ \xi_2^2 \\ \xi_3^2(=\xi_1^3) \\ \vdots \\ \xi_3^{N-1}(=\xi_1^N) \\ \xi_2^N \\ \xi_3^N \end{bmatrix}, \quad \Xi' = \begin{bmatrix} \xi_1' \\ \xi_2' \\ \xi_3' \\ \xi_4' \\ \xi_5' \\ \vdots \\ \xi_{2N-1}' \\ \xi_{2N}' \\ \xi_{2N+1}' \end{bmatrix} = \begin{bmatrix} \xi_1^{1'} \\ \xi_2^{1'} \\ \xi_3^{1'}(=\xi_1^{2'}) \\ \xi_2^{2'} \\ \xi_3^{2'}(=\xi_1^{3'}) \\ \vdots \\ \xi_3^{N-1'}(=\xi_1^{N'}) \\ \xi_2^{N'} \\ \xi_3^{N'} \end{bmatrix},$$

$$\Xi'' = \begin{bmatrix} \xi_1'' \\ \xi_2'' \\ \xi_3'' \\ \xi_4'' \\ \xi_5'' \\ \vdots \\ \xi_{2N-1}'' \\ \xi_{2N}'' \\ \xi_{2N+1}'' \end{bmatrix} = \begin{bmatrix} \xi_1^{1''} \\ \xi_2^{1''} \\ \xi_3^{1''}(=\xi_1^{2''}) \\ \xi_2^{2''} \\ \xi_3^{2''}(=\xi_1^{3''}) \\ \vdots \\ \xi_3^{N-1''}(=\xi_1^{N''}) \\ \xi_2^{N''} \\ \xi_3^{N''} \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ \vdots \\ G_{2N-1} \\ G_{2N} \\ G_{2N+1} \end{bmatrix} = \begin{bmatrix} g_1^1 \\ g_2^1 \\ g_3^1+g_1^2 \\ g_2^2 \\ g_3^2+g_1^3 \\ \vdots \\ g_3^{N-1}+g_1^N \\ g_2^N \\ g_3^N \end{bmatrix}.$$

and

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ \vdots \\ F_{2N-1} \\ F_{2N} \\ F_{2N+1} \end{bmatrix} = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 + f_1^2 \\ f_2^2 \\ f_3^2 + f_1^3 \\ \vdots \\ f_3^{N-1} + f_1^N \\ f_2^N \\ f_3^N \end{bmatrix}.$$

Particular, setting $h_e = h$, the Ψ, Φ can be calculated as

$$\Psi = \frac{h}{30} \begin{bmatrix} 4 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & 8 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & 16 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 8 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 8 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 16 & 2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & 4 \end{bmatrix},$$

and

$$\Phi = \frac{1}{3h} \begin{bmatrix} 7 & -8 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -8 & 16 & -8 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -8 & 14 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 14 & -8 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -8 & 16 & -8 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -8 & 7 \end{bmatrix}.$$

Then by (18), the overall equation in the vector form can be expressed as

$$\Psi \Xi'' + \alpha \Psi \Xi' + \beta \Phi \Xi + \delta G = F. \tag{19}$$

We choose Gauss–Legendre quadrature to integrate the nonlinear terms involving absolute functions. Given the function $r(x)$ and an integer m, we can find a set of weights r_j and abscissas p_j such that the approximation

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{j=1}^{m} r_j f(p_j)$$

Table 1
Gauss–Legendre quadrature points and weights

m	Points p_j	Weights r_j
6	±0.2386191861	0.4679139346
	±0.6612093865	0.3607615730
	±0.9324695142	0.1713244924

is exact if $f(x)$ is a polynomial. The weights are $r_j = \frac{2}{(1-p_j^2)[v_m'(p_j)]^2}$, where $v_m(x)$ are Legendre polynomials and the abscissas for quadrature order m are given by the roots of the Legendre polynomials $v_m(x)$. The points p_j and weights r_j on interval $[-1,1]$ are given in Table 1.

To apply Gauss–Legendre integrate method to the components of g_i, we divide the element interval $[x_e, x_{e+1}]$ into $m=6$ points, by normalized the coordination from $[-1,1]$ to $[x_e, x_{e+1}]$ to obtain the abscissas $p_1^e, p_2^e, \ldots, p_m^e$ on $[x_e, x_{e+1}]$ and weights $r_1, r_2, \ldots, r_m$. Thus, by (17), G is approximated by the new function $\hat{G}$, its components are given in below.

1-st component

$$\hat{g}_1 = \sum_{j=1}^{m} \left|\xi_1^1(t)\psi_1^1(p_j^1)\right|^{\gamma} \xi_1^1(t)\psi_1^1(p_j^1)\psi_1^1(p_j^1)r_j.$$

2-nd component

$$\hat{g}_2 = \sum_{j=1}^{m} \left|\xi_2^1(t)\psi_2^1(p_j^1)\right|^{\gamma} \xi_2^1(t)\psi_2^1(p_j^1)\psi_2^1(p_j^1)r_j.$$

3-rd component

$$\begin{aligned}\hat{g}_3 &= \sum_{j=1}^{m} \left|\xi_3^1(t)\psi_3^1(p_j^1)\right|^{\gamma} \xi_3^1(t)\psi_3^1(p_j^1)\psi_3^1(p_j^1)r_j \\ &\quad + \sum_{j=1}^{m} \left|\xi_1^2(t)\psi_1^2(p_j^2)\right|^{\gamma} \xi_1^2(t)\psi_1^2(p_j^2)\psi_1^2(p_j^2)r_j\end{aligned}$$

and so on, till $(2N-1)$ *component*

$$\begin{aligned}\hat{g}_{2N-1} &= \sum_{j=1}^{m} \left|\xi_3^{N-1}(t)\psi_3^{N-1}(p_j^{N-1})\right|^{\gamma} \xi_3^{N-1}(t)\psi_3^{N-1}(p_j^{N-1})\psi_3^{N-1}(p_j^{N-1})r_j \\ &\quad + \sum_{j=1}^{m} \left|\xi_1^{N}(t)\psi_1^{N}(p_j^{N})\right|^{\gamma} \xi_1^{N}(t)\psi_1^{N}(p_j^{N})\psi_1^{N}(p_j^{N})r_j.\end{aligned}$$

$2N$ *component*

$$\hat{g}_{2N} = \sum_{j=1}^{m} \left|\xi_2^N(t)\psi_2^N(p_j^N)\right|^{\gamma} \xi_2^N(t)\psi_2^N(p_j^N)\psi_2^N(p_j^N)r_j.$$

$2N+1$ *component*

$$\hat{g}_{2N+1} = \sum_{j=1}^{m} \left|\xi_3^N(t)\psi_3^N(p_j^N)\right|^{\gamma} \xi_3^N(t)\psi_3^N(p_j^N)\psi_3^N(p_j^N)r_j.$$

The external input F can be represented by

$$F = \begin{bmatrix} \int_{x_1}^{x_2} g(t)\psi_1^1(x)\,\mathrm{d}x \\ \int_{x_1}^{x_2} g(t)\psi_2^1(x)\,\mathrm{d}x \\ \int_{x_1}^{x_2} g(t)\psi_3^1(x)\,\mathrm{d}x + \int_{x_3}^{x_2} g(t)\psi_1^2(x)\,\mathrm{d}x \\ \int_{x_2}^{x_3} g(t)\psi_2^2(x)\,\mathrm{d}x \\ \int_{x_2}^{x_3} g(t)\psi_3^2(x)\,\mathrm{d}x + \int_{x_3}^{x_2} g(t)\psi_1^3(x)\,\mathrm{d}x \\ \vdots \\ \int_{x_{N-2}}^{x_{N-1}} g(t)\psi_3^{N-1}(x)\,\mathrm{d}x + \int_{x_{N-1}}^{x_N} g(t)\psi_1^N(x)\,\mathrm{d}x \\ \int_{x_{N-1}}^{x_N} g(t)\psi_2^N(x)\,\mathrm{d}x \\ \int_{x_{N-1}}^{x_N} g(t)\psi_3^N(x)\,\mathrm{d}x \end{bmatrix}. \tag{20}$$

If the inverse of Ψ exists, we can convert Eq. (19) into the form of matrixes and vectors as

$$\begin{pmatrix} \Xi \\ \Xi' \end{pmatrix}' + \begin{pmatrix} 0 & -I \\ \beta\Psi^{-1}\Phi & \alpha I \end{pmatrix}\begin{pmatrix} \Xi \\ \Xi' \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi^{-1}F \end{pmatrix} - \begin{pmatrix} 0 \\ \delta\Psi^{-1}\hat{G}(\Xi) \end{pmatrix}. \tag{21}$$

By introducing

$$\mathbf{\Sigma} = \begin{pmatrix} \Xi \\ \Xi' \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 0 & -I \\ \beta\Psi^{-1}\Phi & \alpha I \end{pmatrix}$$

and

$$\mathbf{F} = \begin{pmatrix} 0 \\ \Psi^{-1}F \end{pmatrix}, \quad \mathbf{G}(\mathbf{\Sigma}) = \begin{pmatrix} 0 \\ \delta\Psi^{-1}\hat{G}(\Xi) \end{pmatrix}.$$

(21) becomes

$$\mathbf{\Sigma}' + \mathbf{M}\mathbf{\Sigma} = \mathbf{F} - \mathbf{G}(\mathbf{\Sigma}), \tag{22}$$

which is a first order vector ordinary differential equation. To solve (22), we use the fourth order Runge–Kutta method (cf. [22]). Once the $\xi_i^e(t)$, $i = 1, 2, 3$ are known for $e = 1, 2, \ldots, N$, we can obtain the numerical solution as $y_h = \sum_{e=1}^{N}\sum_{i=1}^{3}\xi_i^e(t)\psi_i^e(x)$ on domain $[0, T] \times (0, l)$.

5. Convergence of the solution

In this section, we consider the convergence of the scheme proposed in Section 4.

Theorem 8. *The numerical solution* $y_h = \sum_{e=1}^{N}\sum_{i=1}^{3}\xi_i^e(t)\psi_i^e(x)$ *converges to the weak solution* y *as* $N \to \infty$.

Proof. We consider the orthogonal basis of H, $\{w_j\}_{j=1}^{\infty}$ consisting of eigenfunctions of A. The interpolation function $\psi_i^e \in V \subset H$ can be represented by the linear combination of $\{w_1, w_2, \ldots, w_m\}$ as $\psi_i^e = \sum_{j=1}^{m}\alpha_j^{e,i}w_j$. Then the global approximate solution can be expressed by

$$y_h = \sum_{j=1}^{m} g_{jm}(t)w_m,$$

where $g_{jm}(t) = \sum_{e=1}^{N}\sum_{i=1}^{3}\xi_i^e(t)\alpha_j^{e,i}$. Refer to [15] and by the existence theorem of weak solution, it is easy to verify that $\{y_h\}$ is bounded in $L^\infty(0,T;V)$ and $\{y_h'\}$ is bounded in $L^\infty(0,T;H)$. Therefore, by the Rellich's extraction theorem, we can find a subsequence of $\{y_h\}$, denotes by itself, and find $z \in L^\infty(0,T;V) \subset L^2(0,T;V)$, $z' \in L^\infty(0,T;H) \subset L^2(0,T;H)$ such that

$$y_h \to z \quad \text{weakly} \ * \ \text{in} \ L^\infty(0,T;V) \quad \text{and weakly in} \ L^2(0,T;V), \tag{23}$$

$$y_h' \to z' \quad \text{weakly} \ * \ \text{in} \ L^\infty(0,T;H) \quad \text{and weakly in} \ L^2(0,T;H). \tag{24}$$

By the classical compactness theorem (cf. [1,19]) the conditions (23) and (24) imply

$$y_h \to z \quad \text{strongly in} \ L^2(0,T;H). \tag{25}$$

Then by the well known theorem on strong convergence and (25), we can extract a subsequence of y_h, denote again by y_h, such that

$$y_h(t,x) \to z(t,x) \quad \text{a.e. in} \ [0,T] \times (0,l).$$

In fact, we can prove that $z(t,x)$ is the weak solution $y(t,x)$ of (14) via uniqueness (cf. [15]). This completes the proof of Theorem 8. $\square$

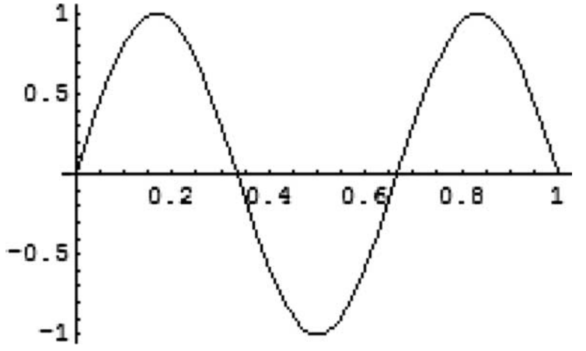


Fig. 2. Initial function $y_0 = \sin(3\pi x)$.

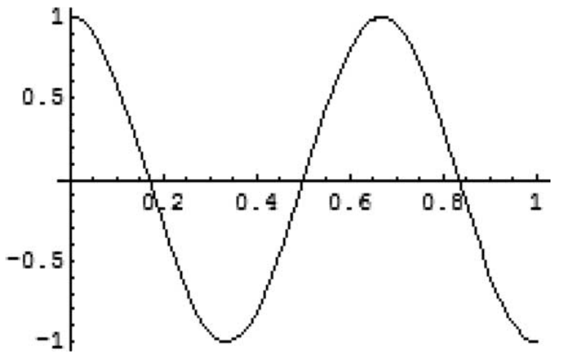


Fig. 3. Initial function $y_1 = \cos(3\pi x)$.

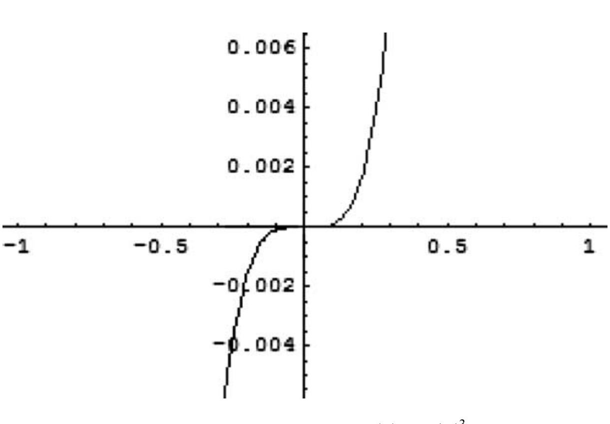


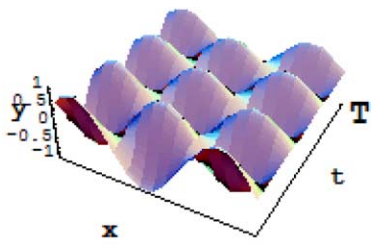
Fig. 4. Nonlinear input $g(y) = y|y|^3$.

6. Numerical experiments

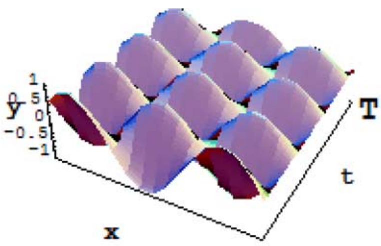
In this section, we give the simulation based on the numerical solution given in Section 4, we obtain 3D graphics of numerical solution for the system with $f = 0$, taking the initial function $y_0(x) = \sin(3\pi x)$ (cf. Fig. 2) and $y_1(x) = \cos(3\pi x)$ (cf. Fig. 3).

Example 6.1. In the simulation given below, we suppose $\Omega = (0, 1)$, $\alpha = 1.0$, $\beta = 1.0$, $t_0 = 0.0$, $T = 1.0$, $\Delta t = 0.01$, $\gamma = 3$, i.e., $g(y) = \delta|y|^3 y$ (cf. Fig. 4).

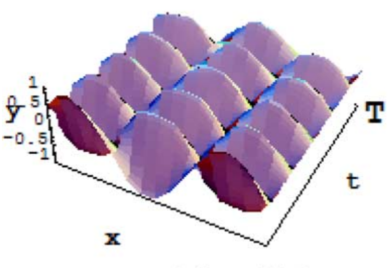
Taking $\delta > 0$ and changing the value of δ, the numerical solution are shown in figures (a)–(d) below:



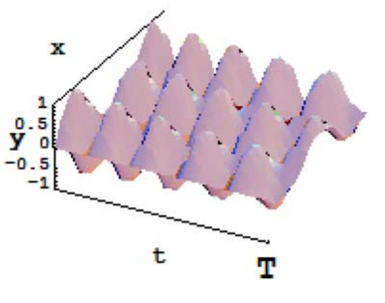
(a) $\gamma = 3, \delta = 0.0$



(b) $\gamma = 3, \delta = 200.0$



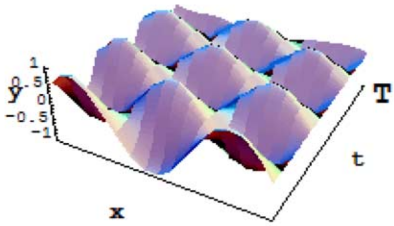
(c) $\gamma = 3, \delta = 1000.0$



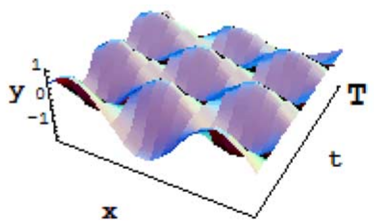
(d) $\delta = 1000.0$,ViewPoint->$\{10, 5, 6\}$

In above graphics, we find the frequency of wave increases with increasing δ and numerical solution is bounded in domain $[0, T] \times (0, l)$. This simulation result illustrates Theorem 7 as $\gamma > 1$, also refer the theoretical results in [15].

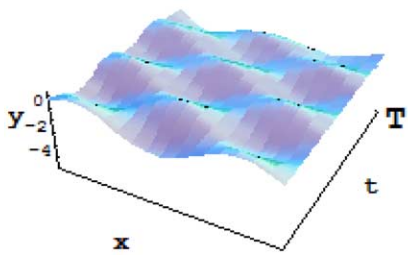
On the other hand, if we take $\delta < 0$ and enlarge the value δ, the graphics of numerical solution can be seen in figures (e)–(h) below:



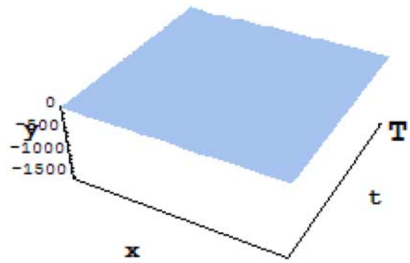
(e) $\gamma = 3, \delta = -100.0$



(f) $\gamma = 3, \delta = -113.0$



(g) $\gamma = 3, \delta = -115.0$



(h) $\gamma = 3, \delta = -116.0$

We found that the frequency of wave decays and the numerical solution becomes unbounded in the domain $[0, T] \times [0, l]$. This explains Theorem 6 as $\gamma > 1$ (cf. [15]).

Let $x = 0.5$ be fixed, Fig. 5 shows the change of the numerical solution with increasing the time on $[0, T]$.

It showed the wave phenomena with enlarging the coefficients of nonlinear inputs as $\gamma > 1$, it is also observed that the solution approaches to the stable state as time increases.

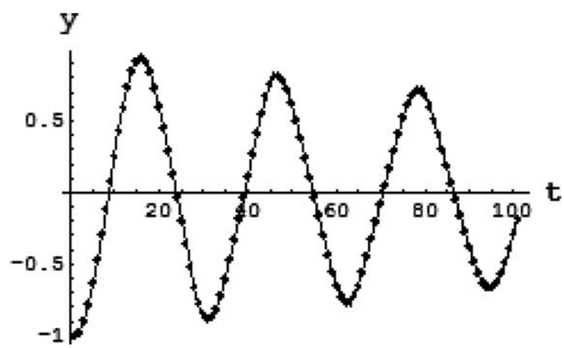
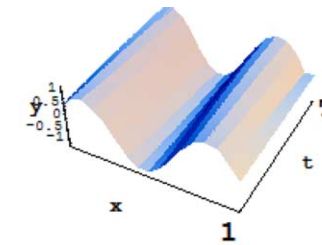


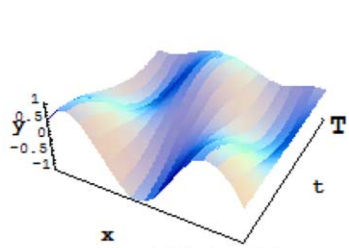
Fig. 5. $x = 0.5$, $\gamma = 3$, $\delta = 100.0$.

Example 6.2. In the simulations given below, we suppose $\Omega = (0,1)$, $\alpha = 1.0$, $\beta = 0.0001$, $t_0 = 0.0$, $T = 1.0$, $\Delta t = 0.05$. The initial functions $y_0(x) = \sin(3\pi x)$ and $y_1(x) = \cos(3\pi x)$ are shown in Figs. 2 and 3. The exponent of nonlinear term $\gamma = 0.001$ and $g(y) = \delta y|y|^{\gamma}$ are shown in Fig. 6.

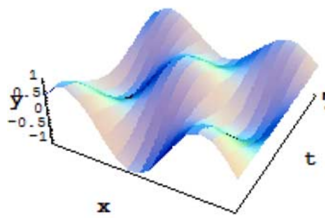
We show the numerical solution with change of nonlinear term, the constant δ is the parameter to be adjusted, see figures (a)–(h) below:



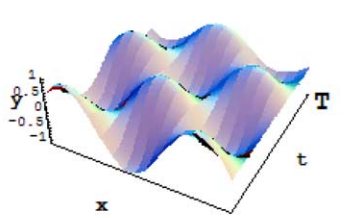
(a) $\gamma = 0.001, \delta = 0.0$



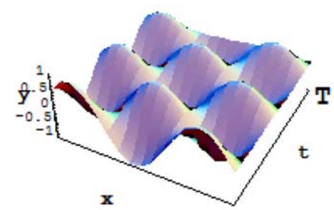
(b) $\gamma = 0.001, \delta = 10.0$



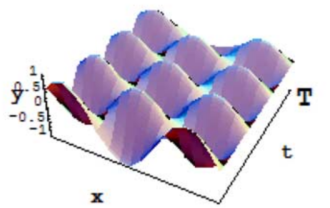
(c) $\gamma = 0.001, \delta = 20.0$



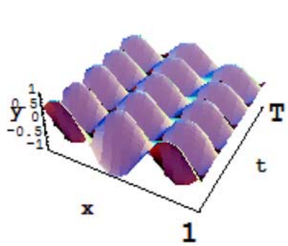
(d) $\gamma = 0.001, \delta = 50.0$



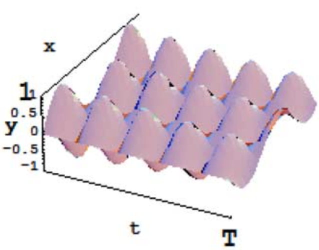
(e) $\gamma = 0.001, \delta = 100.0$



(f) $\gamma = 0.001, \delta = 200.0$



(g) $\gamma = 0.001, \delta = 400.0$



(h) $\delta = 400.0$,ViewPoint->{10,5,6}

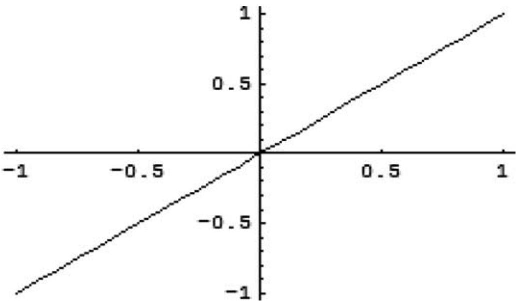


Fig. 6. Nonlinear input $y|y|^{\gamma}$, $\gamma = 0.001$.

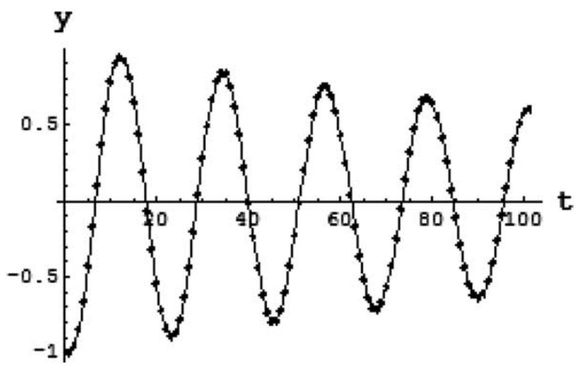
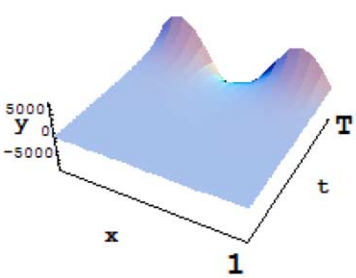


Fig. 7. $x = 0.5$, $\gamma = 0.001$, $\delta = 400.0$.

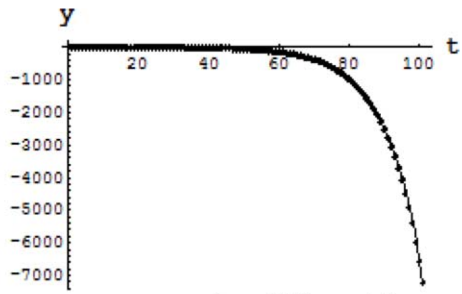
Let $x = 0.5$ be fixed, the numerical solution with respect to t is shown in Fig. 7.

We found an obviously frequency change of the wave as $\delta > 0$, and the numerical solution tends to the steady state as time t increases. This case is another form of Theorem 7 as $\gamma < 1$.

If we take $\delta < 0$, the feature of numerical solution is shown in figure (i) below:



(i) $\gamma = 0.001, \delta = -50.0$



(j) $\delta = -50.0, x = 0.5$

Let x be fixed, the numerical solution with respect to time t is shown in figure (j). Here $x = 0.5$, $t_0 = 0.0$, $T = 1.0$ and $\Delta t = 0.01$.

It can be found that the graphics is not a regular wave phenomena and the numerical solution is unbounded on the domain $(0, T) \times (0, l)$. This is another evidence of Theorem 6 as $\gamma < 1$.

The graphics in Example 6.2 showed the wave change with increasing parameter δ as $\gamma < 1$. One can find the numerical solution possessing wave phenomena.

7. Conclusions

In this paper, we studied the numerical solution using variational method and finite element approximation. A semi-discrete algorithm has been developed for the numerical solution of damped nonlinear Klein–Gordon equation, for two difference exponents $\gamma > 1$ and $\gamma < 1$ of nonlinear input, some numerical experiments were calculated to verify the effectiveness of the scheme. The wave graphics of the dynamics systems have been provided for the various physics parameters. Meanwhile, we can view the tendency with respect to the time. The results revealed the behavior of solution of nonlinear Klein–Gordon equation.

Comparing the research results of the equations on physical field (cf. [10,16]), the periodic wave, narrow kink, oscillating kink phenomena can be found in our simulation. It verifies the efficient of the numerical treatment to damped nonlinear Klein–Gordon equation.

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