第34卷 第3期 吉林大学学报(工学版) Vol.34 No.3
2004年7月 Journal of Jilin University (Engineering and Technology Edition) July 2004

文章编号:1671-5497(2004)03-0408-04

# Stability of switched linear systems via cascading

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**Abstract:** In order to investigate the problem of quadratic stability of switched linear systems via cascading, all the real invariant subspaces of a given linear system were investigated, and the result was used to provide comparable cascading form of switching models. Using the common cascading form, a common quadratic Lyapunov function (CQLF) can be found by the set of CQLF of diagonal blocks.

**Key words:** switching linear system; quadratic stability; CQLF

**CLC Number:** O231.1 **Document Code:** A

## 1 Introduction

In recent years, one important topic in investigating switched systems is the stability[1,2]. A very natural way, but not necessary, is to find a CQLF for all switching models. Such a Lyapunov function suffices for the stability of systems under arbitrary switchings. For linear switching models the quadratic Lyapunov function plays an important role.

**Definition 1** Consider a switched linear system

$$\dot{x} = A_{\sigma(t)}x, \qquad x \in \mathbf{R}^n \tag{1}$$

where $\sigma(t):[0,\infty)\rightarrow\Lambda$ is a measurable mapping, and $\Lambda=\{1,\cdots,N\}$. The system (1) is said to be quadratically stable if there is a positive definite matrix $P>0$, such that

$$PA_i + A_i^{\mathrm{T}}P < 0 \qquad (i \in \Lambda) \tag{2}$$

If (2) holds, we say that (1) shares a CQLF. Many different methods have been used to solve (2)[3~5]. Recently, a necessary and sufficient condition for the existence of CQLF for a set of stable matrices was presented[6]. In fact, [6] provides a numerical method for the existence. Only when $n=2$, it becomes an easily verifiable, necessary and sufficient condition. Stabilization for switched system is an even harder problem, for planar switched linear systems a necessary and sufficient condition was given in [7]. For $n>2$, the problem remains open. An useful result is given in the following.

**Theorem 1**[6] Let $A_i, i\in\Lambda$ be a set of Hurwitz matrices with same block upper triangular structure, i.e.

$$A_i = \begin{pmatrix} A_i^{11} & A_i^{12} & \cdots & A_i^{1s} \\ 0 & A_i^{22} & \cdots & A_i^{2s} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_i^{ss} \end{pmatrix} \qquad i \in \Lambda \tag{3}$$

**Accepted date:** 2004-02-16.

**Foundation item:** Supported by the Chinese National Natural Science Foundation(60343001, 60274010).

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where $A_i^{kk}(k=1,\cdots,s)$ are $n_k\times n_k$ matrices. Then $A_i$ share a CQLF, iff $A_i^{kk}$ share a CQLF for all $k=1,\cdots,s$.

The purpose of this paper is to investigate when the switched linear system (1) can have the cascade form (3). Then Theorem 2 can be used to test the quadratic stability and stabilization of the system.

The paper is organized as follows. Section 2 considers the relationship between the cascade form and invariant subspaces. Section 3 provides a complete description of the invariant subspaces of a given linear system. Section 4 investigates the quadratic stability by using cascading realization. Some examples are included. Section 5 is the conclusion . All the proofs of proposition and Theorems are omitted because of the space.

## 2 Cascade form vs invariant subspace

In this section, it is shown that for a given matrix its cascade form is closely related to its invariant sub-spaces, which are defined as follows:

**Proposition 1** $A_i, i\in\Lambda$ can be converted into the cascade form (3), iff there exists a quasi-flag such that $AV_j\subset V_j$ for all $j=1,\cdots,s-1$.

**Definition 2** For a linear system $\dot{x}=Ax, x\in\mathbf{R}^n$, a subspace $v\subset\mathbf{R}^n$ is called $A$-invariant if $AV\subset V$

**Definition 3** A sequence of nested subspace of $\mathbf{R}^n$, $V_1\subset\cdots\subset V_{s-1}\subset V_s=\mathbf{R}^n$ is called a quasi-flag. Next it is considered when $A_i$ in system(1) can be converted into the cascade form (3) simultaneously.

## 3 Structure of invariant subspaces

In this section we investigate the structures of all real invariant subspaces of a given matrix $A$. Suppose the minimum polynomial of $A$ has the form as the following:

$$m(\lambda)=(\lambda-\lambda_1)^{l_1}\cdots(\lambda-\lambda_k)^{l_k}(\lambda-\gamma_1)^{c_1}(\lambda-\overline{\gamma}_1)^{c_1}\cdots(\lambda-\gamma_s)^{c_s}(\lambda-\overline{\gamma}_s)^{c_s}\tag{4}$$

where $\lambda_t, t=1,\cdots,k;\gamma_j=\alpha_j+\beta_j i, j=1,\cdots,s$, are two groups of different real and complex eigenvalues of $A$ respectively.

**Lemma 1**[8] The whole space $\mathbf{R}^n$ can be decomposed into several kernel spaces as

$$\mathbf{R}^n=V_1\oplus\cdots\oplus V_k\oplus W_1\oplus\cdots\oplus W_s\tag{5}$$

where $V_i=\ker(A-\lambda_t I)^{l_t}, t=1,\cdots,k,\ W_j=\ker(A^2-2\alpha A+(\alpha^2+\beta^2)I), j=1,\cdots,s$

**Proposition 2** If $H$ is $A$-invariant, then $H=H\cap V_1\oplus\cdots\oplus H\cap V_k\oplus H\cap W_1\oplus\cdots\oplus H\cap W_s$

Using this Proposition, the supposition that matrix $A$ has only one real eigenvalue or a pair of conjugate complex eigenvalues is reasonable.

(A1) Matrix $A_{n\times n}$ has a real eigenvalue $\lambda$ and $T^{-1}TA=\mathrm{diag}\{J_{s_1\times s_1}(\lambda)\cdots J_{s_m\times s_m}(\lambda)\}$, where $T=[e_1^1\cdots e_1^{s_1},\cdots,e_m^1\cdots e_m^{s_m}]$.

Now the author is ready to find any $t$-dimensional $A$-invariant subspace, say $H$. The auther denot $s=\max\{s_i\mid i=1,\cdots,m\}$, Using $Z_+$ for the set of positive integers, the auther defines

$$K_t=\{k=(k_1,\cdots,k_\tau)\in Z_+^\tau\mid \tau\leqslant m,\sum_{j=1}^{\tau}k_j=t, 0<k_j\leqslant k_{j+1}\leqslant s\}\tag{6}$$

**Proposition 3** Assume (A1) holds for $A_{n\times n}$, then for each $k\in K_t$ there exists a $t$-dimensional subspace $H$ which can be decomposed into a direct sum of $z(\xi_j, A)$, if $[c_{1,k_j}^j,\cdots,c_{m,k_j}^j]$, $j=1,\cdots,\tau$, are linearly independent, where it is assumed a set of vectors in $H$ can be expressed as $\xi_j=\sum_{i=1}^{m}\sum_{l=1}^{k_j}c_{i,l}^j e_i^l$ and

$Z(\xi_j, A) = \text{span}\{\xi_j, A\xi_j, \cdots, A^{k_j-1}\xi\}$.

(A2) Matrix $A_{2n\times 2n}$ has a pair of conjugate complex eigenvalues $\alpha \pm \beta i$, and under basis $T = [e_1^1, \cdots, e_1^{2s_1}, \cdots, e_m^1, \cdots, e_m^{2s_m}]$, $A$ can be expressed as classical form $T^{-1}AT = \text{diag}\{J_{2s_1\times 2s_1}, \cdots, J_{2s_m\times 2s_m}\}$. The auther proposes another form for convenience, which is $\widetilde{T}^{-1}A\widetilde{T} = \text{diag}\{\widetilde{J}_{2s_1\times 2s_1}, \cdots, \widetilde{J}_{2s_m\times 2s_m}\}$. Proving the following relationship, the auther denotes $C = A - \lambda I$, $B = A^2 - 2\alpha A + (\alpha^2 + \beta^2)I$.

$$\widetilde{T} = [\eta_1^1 \cdots \eta_1^{2s_1}, \cdots, \eta_m^1 \cdots \eta_m^{2s_1}] \quad \eta_i^{2s_i} = e_i^{2s_i} \quad \eta_i^{2s_i-1} = \beta^{-1}Ce_i^{2s_i} \quad \eta_i^{2s_i-2} = \beta^{-1}Be_i^{2s_i} \quad \eta_i^{2s_i-3} = \beta^{-2}CBe_i^{2s_i}$$

$$\eta_i^{2s_i-4} = \beta^{-2}B^2e_i^{2s_i} \quad \eta_i^2 = \beta^{-(n-1)}B^{n-1}e_i^{2s_i} \quad \eta_i^1 = \beta^{-n}B^{n-1}Ce_i^{2s_i}$$

$$D = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \widetilde{E} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_{2s_i\times 2s_i} = \begin{bmatrix} D & E & 0 & \cdots & 0 \\ 0 & D & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & E \\ 0 & 0 & 0 & \cdots & D \end{bmatrix} \qquad \widetilde{J}_{2s_i\times 2s_i} = \begin{bmatrix} D & \widetilde{E} & 0 & \cdots & 0 \\ 0 & D & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \widetilde{E} \\ 0 & 0 & 0 & \cdots & D \end{bmatrix}$$

Similar to Proposion 3, we have the following result, here $K_t$ is defined as (6).

**Proposition 4** Assume (A2) holds for $A_{2n\times 2n}$. Then for each $k \in K_t$ there exists a $2t$-dimensional subspace $H$ which can be decomposed into a direct sum of $z(\xi_j, A)$, if $[c^j_{1,2k_j} \cdots c^j_{m,2k_j}, -c^j_{1,2k_j-1} \cdots -c^j_{1,2k_j-1}]$, $[c^j_{1,2k_j-1} \cdots -c^j_{m,2k_j-1}, c^j_{1,2k_j} \cdots c^j_{1,2k_j}]$ $(j = 1, \cdots, \tau)$ are linearly independent, where we assume a set of vectors in $H$ can be expressed as $\xi_j = \sum_{i=1}^{m}\sum_{U=1}^{2k} c^j_{i,l}\eta_i^l$ and $z(\xi_j, A) = \text{span}\{\xi_j, A\xi_j, \cdots, A^{2k_j-1}\xi_j\}$.

Using Propositions 2, 3 and 4, we can put blocks of different eigenvalues together. Denote by $Z_0$ the set of non-negative integers and define an index set as

$$M_t = \{\mu = (\mu_1 \cdots \mu_k, \mu_{k+1} \cdots, \mu_{k+s}) \in Z_0^{k+s} \mid, \sum_{j=1}^{k}\mu_j + 2\sum_{l=k+1}^{k+s}\mu_l = t\}$$

**Theorem 2** Assume a matrix $A$ has its minimum polynomial as (4) and $\mathbf{R}^n$ is decomposed as (5). Then a $t$-dimensional real $A$-invariant subspace $H$ is a direct sum of $H_i (i = 1, \cdots, k+s)$. Each set of $\{H_i\}$ is generated from one $\mu \in M_t$. That is, a $\mu_i$-dimensional subspace $H_i (i = 1, \cdots, k)$ are obtained from $V_i$ and a $2\mu_{k+i}$-dimensional subspace $H_{k+j} (j = 1, \cdots, s)$ are obtained from $W_j$.

## 4 Example

Find the CQLF of $A_i, i = 1, 2$.

$$A_1 = \begin{bmatrix} 10 & -10.5 & 7 & -5 \\ 10/3 & -13/3 & 10/3 & -2 \\ 46/3 & -43/3 & 19/3 & -6 \\ 40 & -37.5 & 20 & -17 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -7 & 3 & -3 & 2 \\ 0 & -4 & 0 & 0 \\ -8 & 8 & -9 & 4 \\ -55/3 & 55/3 & -40/3 & 6 \end{bmatrix}$$

Step1: with straightforward calculating, $A_i$ has Jordan form as follows:

$$T_1^{-1}A_1T_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad T_2^{-1}A_2T_2 = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix},$$

where $$T_1 = \begin{bmatrix} 1 & 3 & 9 & 5 \\ 0 & 2 & 8 & 2 \\ 2 & 3 & 5 & 6 \\ 5 & 5 & 10 & 16 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & -4 \\ 5 & 0 & 0 & -9 \end{bmatrix}$$

Step2: Compute all invariant subspaces of $A_i$, Denote $T_1 := [e_1 \cdots e_4]$, $T_2 := [\eta_1 \cdots \eta_4]$.

(1) 1-dimensional invariant subspace of $A_1$, $V_1^1 = \text{span}\{a_1 e_1 + b_1 e_3\}$, $V_1^2 = \text{span}\{e_4\}$

1-dimensional invariant subspace of $A_2$, $U_1^1 = \text{span}\{c_1 \eta_3 + d_1 \eta_4\}$, $U_1^2 = \text{span}\{\eta_1\}$

(2) 2-dimensional invariant subspaces of $A_1$, $V_2^1 = \text{span}\{e_1, e_3\}$, $V_2^2 = \text{span}\{e_2 + be_1 + ce_3, e_1\}$, $V_2^3 = V_1^1 \oplus V_1^2$

2-dimensional invariant subspaces of $A_2$, $U_2^1 = \text{span}\{\eta_1, \eta_2\}$, $U_2^2 = \text{span}\{\eta_3, \eta_4\}$, $U_2^3 = U_1^1 \oplus U_1^2$

Step3: By solving equation, if we choose $a_1 = 1, b_1 = 0, a_2 = -1, b_2 = -1$ and $c_1 = 1, d_1 = 0$, then under $T = [\eta_1, \eta_3, \eta_2, \eta_4]$, $A_1$ and $A_2$ have the same block upper triangle structure.

Step4: Using the method in [1], we can find CQLF $P$ of $T^{-1}A_iT$.

Step5: Back to the original coordinate frame $(T^{-1})^{\mathrm{T}} P (T^{-1})$ is the CQLF of $A_i$.

## 5 Conclusion

This paper provided a systematic method to find all possible real quasi-flags for a given matrix. Using it, the switched linear system can be expressed as a cascading form. Using this form the verification of quadratic stability for the switched system can be simplified a lot. The results will be used to solve the quadratic stabilization problem.

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