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Boundaries of Conditional Quadratic Forms—A Comment on "Stabilization via Static Output Feedback"

D. Cheng and C. F. Martin

***Abstract*— Motivated by the above paper,[1] this note considers the boundaries of a quadratic form with all possible constraints over a given subspace. Essential upper (or lower) bounds are presented provided they exist. It mends a mild incompleteness in the proof of the main result.**

I. INTRODUCTION

It is well known (see, e.g., [2]) that for a real symmetric matrix M

$$\min\sigma(M) \le \frac{x^TMx}{x^Tx} \le \max\sigma(M), x \neq 0 \tag{1}$$

where $\sigma(M)$ is the set of eigenvalues of M.

In the above paper, necessary and sufficient conditions for the existence of a stabilizing static output feedback gain matrix were presented. In the proof of the main result, Theorem 3.1, the following fact was used (for the sake of consistency we use our notations).

Given a real symmetric matrix: $M_{n\times n}$, and a matrix $K_{m\times n}$ with $\text{rank}(K) = r < n$. Assume

$$x^T(M)x < 0; \forall x \in Ker(K). \tag{2}$$

Then

$$\alpha = \begin{matrix} sup \\ x \end{matrix} \frac{x^TMx}{x^TK^TKx} < \infty, x \notin Ker(K). \tag{3}$$

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D. Cheng is with Gifford Fong Associates, Walnut Creek, CA 94596 USA.
C. F. Martin is with the Department of Mathematics, Texas Tech University, Lubbock, TX 79409 USA.

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A mild incompleteness in the proof[1] is: They did not claim and prove that the α (defined in (3.4), which is the same as in (3)), is upper bounded, i.e., $\alpha < +\infty$. It is essential for constructing R ($R^{-1} \ge \alpha I$). It was pointed by a nominated reviewer that this boundedness is also assumed in [2] without proof. From the following discussion one sees that this fact is not trivial. We call it the problem of boundaries of conditional quadratic forms. It can be considered as a generalization of (1), because when $\dim(K) = 0$, our results in this note will coincide with (1).

In the next section we prove (3) by giving the essential upper bound, which may be of independent interest. Then in Section III we discuss all other constraints.

II. MAIN RESULT

Let the matrices $M_{n\times n}$, $K_{m\times n}$, with $\text{rank}(K) = r < n$, be as above. Then there exists a linear transformation Φ such that

$$K\Phi = (S \quad | \quad 0)$$

where S is the first r columns, and thus S has full rank. It follows that S^TS is a positive definite matrix, so we can define a positive definite E as

$$E = (S^TS)^{1/2} > 0. \tag{4}$$

It is easy to prove that (2) is equivalent to the fact that after the linear transformation Φ, M has the following form

$$\Phi^TM\Phi = \begin{pmatrix} A & B \\ B^T & -Q \end{pmatrix}. \tag{5}$$

Then we define a characteristic matrix C as

$$C = E^{-1}(A + BQ^{-1}B^T)E^{-1}. \tag{6}$$

Using the above notations, we can prove the following theorem.

Theorem 1: Under condition (2), we have the essential upper bound as

$$\begin{aligned} &\begin{matrix} sup \\ x \end{matrix} \frac{x^TMx}{x^TK^TKx}, x \notin Ker(K) \\ &= \max\sigma(C). \end{aligned} \tag{7}$$

Proof: It is clear that $x \in Ker(K)$, if and only if

$$y = \Phi^{-1}x = \begin{pmatrix} 0 \\ y_2 \end{pmatrix}.$$

Since Q is positive definite, $Q^{1/2} > 0$ is well defined. Using (5), a straightforward computation shows that

$$\begin{aligned} &\begin{matrix} sup \\ x \end{matrix} \frac{x^TMx}{x^TK^TKx}, x \notin Ker(K) \\ &= \begin{matrix} sup \\ y \end{matrix} \frac{y^T\begin{pmatrix} A & B \\ B^T & -Q \end{pmatrix}y}{y^T\begin{pmatrix} S^TS & 0 \\ 0 & 0 \end{pmatrix}y} \\ &= \begin{matrix} sup \\ y \end{matrix} \frac{y_1^T(A + BQ^{-1}B^T)y_1 - \| Q^{1/2}y_2 - Q^{-1/2}B^Ty_1 \|^2}{y_1^TS^TSy_1} \\ &= \begin{matrix} sup \\ y_1 \neq 0 \end{matrix} \frac{y_1^T(A + BQ^{-1}B^T)y_1}{y_1^TS^TSy_1} \end{aligned} \tag{8}$$

[1] A. Trofino-Neto and V. Kucera, *IEEE Trans. Automat. Contr.*, vol. 38, no. 5, pp. 764-765, May 1993.

where

$$y = \Phi^{-1}x = \frac{y_1}{y_2}, \quad y_1 \neq 0.$$

The last equality is obtained by setting $y_2 = Q^{-1}B^T y_1$.

Recalling the definitions of E and C in (4) and (6), we get

$$\sup_{y_1 \neq 0} \frac{y_1^T(A + BQ^{-1}B^T)y_1}{y_1^T S^T S y_1} = \sup_{Z \neq 0} \frac{Z^T C Z}{Z^T Z}, \text{where } Z = Ey_1.$$

From (1), it can be seen that

$$\sup_x \frac{x^T M x}{x^T K^T K x}, x \notin Ker(K) = \max\sigma(C) < +\infty. \tag{9}$$

Since the transformation Φ is not unique, the last thing we have to do is to show that the essential upper bound obtained in (9) is independent of the choice of Φ. Theoretically, essential upper bound is unique. But, we use a particular Φ to get it, and the parameters A, B, C, E, etc. in the expression depend on Φ. So we must show that the upper bound is independent of Φ. Let $\tilde{\Phi} = (\tilde{\phi}_1|\tilde{\phi}_2)$, where Span$(\tilde{\phi}_2) = K^{\perp}$, be another suitable linear transformation, and $\tilde{\Phi} = \Phi T = (\phi_1|\phi_2)T$. Since Span$(\tilde{\phi}_2)$ = Span$(\phi_2) = K^{\perp}$, it follows that

$$T = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}.$$

Now, a straightforward computation shows that the corresponding expressions under the new transformation are

$$\begin{pmatrix} \tilde{S}^T\tilde{S} & 0 \\ 0 & 0 \end{pmatrix} = (K\tilde{\Phi})^T(K\tilde{\Phi}) = T^T(K\Phi)^T(K\Phi)T = \begin{pmatrix} T_1^T S^T S T_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B}^T & -\tilde{Q} \end{pmatrix} = \begin{pmatrix} T_1^T & T_2^T \\ 0 & T_3^T \end{pmatrix}\begin{pmatrix} A & B \\ B^T & -Q \end{pmatrix}\begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}$$
$$= \begin{pmatrix} T_1^T AT_1 + T_2^T B^T T_1 + T_1^T BT_2 - T_2^T QT_2 & T_1^T BT_3 - T_2^T QT_3 \\ T_3^T B^T T_1 - T_3^T QT_2 & -T_3^T QT_3 \end{pmatrix}. \tag{10}$$

Using them, we finally have $\tilde{S}^T\tilde{S} = T_1^T S^T S T_1$, $\tilde{A} + \tilde{B}\tilde{Q}^{-1}\tilde{B}^T = T_1^T(A + BQ^{-1})$.

Therefore, the parameters obtained by the new transformation provide

$$\sup_{y_1 \neq 0} \frac{y_1^T(\tilde{A} + \tilde{B}\tilde{Q}^{-1}\tilde{B}^T)y_1}{y_1^T \tilde{S}^T \tilde{S} y_1} = \sup_{z \neq 0} \frac{Z^T C Z}{Z^T Z}, \text{where } Z = ET_1 y_1 .$$

Q.E.D

III. Generalization

As we mentioned before, conditional quadratic form (3) is a generalization of the famous result (1). So the boundary problem of expression (3) has both theoretical and practical interests. To make (3) meaningful $x \notin$ Ker(K) . Moreover, without constrain on Ker (K) expression (3) has no boundary. (2) is a particular constrain. This section consider all other possible constraints on subspace Ker(K). They may be applied to mini-max problems of quadratic forms.

Case 2: Condition (2) is replaced by

$$x^T(M)x > 0; \forall x \in Ker(K). \tag{11}$$

In this case, replace $-Q$ by Q in (5) and redefine C in (6) accordingly. Then a parallel discussion shows the following corollary.

Corollary 1: Under condition (11), we have

$$\text{sub}_x \frac{x^T M x}{x^T K^T K x}, x \notin \text{Ker}(K) = \min\sigma(C). \tag{12}$$

Case 3: Assume now we have

$$x^T(M)x = 0; \forall x \in \text{Ker}(K). \tag{13}$$

In this case replace Q by zero in (5). Equation (8) becomes

$$\sup_x \frac{x^T M x}{x^T K^T K x}, x \notin \text{Ker}(K) = \sup_y \frac{y_1^T(A)y_1 + 2y_2^T B^T y_1}{y_1^T S^T S y_1}. \tag{14}$$

From (14) one sees the following corollary.

Corollary 2: Under condition (13), if $B \neq 0$

$$\frac{x^T M x}{x^T K^T K x}, x \notin \text{Ker}(K)$$

has neither upper bound nor lower bound. If $B = 0$

$$\min\sigma(E^{-1}ME^{-1}) \leq \frac{x^T M x}{x^T K^T K x} \leq \max\sigma(E^{-1}ME^{-1}), \quad x \notin \text{Ker}(K).$$

Proof: Observe (14). If $B = 0$, the conclusion follows from the standard result (1). If $B \neq 0$, choose

$$y_2 = \mu B^T y_1, k \in R.$$

Letting μ go to either $-\infty$ or $+\infty$, one sees that neither upper bound nor lower bound exist. Q.E.D.

Note that in Case 3 we have from (10) that $B = T_1{}^T BT_3$, and T_1 and T are nonsingular. It is clear that the conclusion in the above corollary is independent of the linear transformation.

Case 4: Assume

$$x^T(M)x \leq 0; \forall x \in \text{Ker}(K). \tag{15}$$

In this case the matrix Q in (5) is positive semi-definite. Then we have the following corollary.

Corollary 3: Under condition (15), if there exists a matrix H with suitable dimension such that $B = HQ$, then

$$\sup_x \left(\frac{x^T M x}{x^T K^T K x}\right) = \max\sigma(E^{-1}(A + HQH^T)E^{-1}), \quad x \notin \text{Ker}(K).$$

Otherwise

$$\frac{x^T M x}{x^T K^T K x}, x \notin \text{Ker}(K)$$

has neither upper bound nor lower bound.

Proof: Let $B = HQ$. Then we can choose

$$T = \begin{pmatrix} I & 0 \\ H^T & I \end{pmatrix}.$$

From (10), new M is block diagonal, and then (8) becomes

$$\sup_x \frac{x^T M x}{x^T K^T K x}, x \notin \text{Ker}(K) = \sup_y \frac{y_1^T(A + HQH^T)y_1 - y_2^T Q y_2}{y_1^T S^T S y_1} \leq \sup_{y_1} \frac{y_1^T(A + HQH^T)y_1}{y_1^T S^T S y_1}.$$

The conclusion follows.

If $B \neq HQ$, the rows of B are not all in $Span\ row\ Q$. Without loss of generality we assume

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}, where\ Q_1 > 0$$

and partition y_2 as

$$y_2 = \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}.$$

Now (8) has the form

$$\frac{x^T M x}{x^T K^T K x}, \quad x \notin \text{Ker}(K)$$
$$\frac{y_1^T (A) y)1 + 2(y_{21}^T, y_{22}^T) B^T y_1 - t_{21}^T Q_1 y_{21}}{y_1^T S^T S y_1}.$$

The condition: "rows of B are not all in $Span\ row\ Q$" implies that By_{22} is not always zero. It is obvious that this term can make the value of the fraction be both positive and negative unbounded.

One can also see from (10) that the condition $B = HQ$ is independent of the linear transformation.

Q.E.D.

Case 5: Assume

$$x^T (M) x \geq 0; \forall x \in \text{Ker}(K). \tag{19}$$

In this case replace $-Q$ by a positive semi-definite Q in (5). Similar to the proof of Case 4, we can show the following.

Corollary 4: Under condition (19), if there exists a suitable dimensional matrix H such that $B = HQ$, then

$$\underset{x}{sub} \left(\frac{x^T M x}{x^T K^T K x} \right) = \min \sigma (E^{-1}(A + HQH^T)E^{-1}),$$
$$x \notin \text{Ker}(K).$$

Otherwise

$$\frac{x^T M x}{x^T K^T K x}, x \notin \text{Ker}(K)$$

has neither upper bound nor lower bound.

IV. Conclusion

In this note we discussed the problem of finding the boundaries of conditional quadratic forms with all possible constraints over a subspace. In all cases, the necessary and sufficient conditions for the existence of upper and/or lower bounds are presented. The essential bounds are obtained whenever they exist.

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On Robust Stability of 2-D Discrete Systems

W.-S. Lu

***Abstract*— This note presents a study on robust stability of two-dimensional (2-D) discrete systems in the Fornasini–Marchesini (F-M) state space setting. A measure of stability robustness of a stable F-M model is defined. Relation of this measure to its counterpart in the Roessor state space and related computational issues are addressed. Three lower bounds of the stability-robustness measure defined are derived using an one-dimensional parameterization approach and a 2-D Lyapunov approach. A numerical example is included to illustrate the main results obtained.**

I. Introduction

In this note, we present a study on robust stability of two-dimensional (2-D) discrete systems under unstructured perturbations. Throughout the concerned system is modeled in the Fornasini-Marchesini (F-M) local state space [1] as

$$x(i+1, j+1) = A_1 x(i, j+1) + A_2 x(i+1, j) \tag{1}$$

where $x(i,j) \in R^{n \times 1}$, A_1, $A_2 \in R^{n \times n}$. Recall that system (1) is asymptotically stable if and only if

$$p(z_1, z_2) \equiv \det(I_n - z_1 A_1 - z_2 A_2) \neq 0 \quad \text{for} \quad (z_1, z_2) \in \bar{U} \tag{2}$$

where $\bar{U} = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1,\}$ [1]. To date, results on robust stability of 2-D discrete systems in a local state-space framework are only available for the Roesser model [2]–[6], and one might attribute this to the lack of a Lyapunov stability theory for the F-M model. The objectives of this note are twofold. First, we propose in Section II a quantitative measured, ν, for the unstructured, stable perturbations of a given stable 2-D F-M system, and derive in Section III a lower bound for ν. Issues on numerical evaluation of the bound obtained and the relation of the proposed stability robustness measure with its counterpart in the Roesser state space will also be addressed. Second, we propose in Section IV a Lyapunov approach to analyzing the robust stability of (1), leading to two lower bounds of ν. The proposed approach makes use of the 2-D Lyapunov equation [7] which is a generalization of the 2-D Lyapunov equation investigated recently by Hinamoto [8]. A numerical example is included in Section V to illustrate the main results of the paper.

In the rest of the paper we write $H > 0$ or $H \geq 0$ to mean that the symmetric matrix H is positive definite or positive semi-definite. For a real matrix $P \geq 0$, one can always write

$$P = U^T \Sigma U$$

where U is orthogonal and $\Sigma = \text{diag}\{\sigma_1, \cdots, \sigma_n\}$ with $\sigma_k \geq 0$, for $1 \leq k \leq n$. If we denote $\Sigma^{1/2} = \text{diag}\{\sigma_1^{1/2}, \cdots, \sigma_n^{1/2}\}$, then

$$P = P^{T/2} P^{1/2}$$

where

$$P^{1/2} = \Sigma^{1/2} U \quad \text{and} \quad P^{T/2} = (P^{1/2})^T.$$

Such a $P^{1/2}$ is called the nonsymmetric square root of P. The largest and smallest singular value of matrix H is denoted by $\bar{\sigma}(H)$ and

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The author is with the Department of Electrical and Computer Engineering, University of Victoria, Victoria, BC, Canada V8W 3P6.

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