then the conclusion holds for all but a finite number of values of β. If $\hat{H}(s_1, \cdots, s_\infty)$ is a polynomial system and (17) is a minimal bilinear realization, then the conclusion holds for every value of β.

Proof: By Theorem 3.2, existence of the corresponding operating points (β, y_β) and $(\beta, x_\beta, y_\beta)$, and of $(A + D\beta)^{-1}$ is guaranteed for β sufficiently small. From (17) and (20), $L_\beta(s)$ can be written as

$$\begin{aligned}L_\beta(s) &= c(sI-A)^{-1}b+\beta[c(sI-A)^{-1}D(sI-A)^{-1}b\\&\quad+c(sI-A)^{-1}D(-A)^{-1}b]\\&\quad+\beta^2[c(sI-A)^{-1}D(sI-A)^{-1}D(sI-A)^{-1}b\\&\quad+c(sI-A)^{-1}D(sI-A)^{-1}D(-A)^{-1}b\\&\quad+c(sI-A)^{-1}D(-A)^{-1}D(-A)^{-1}b]\\&\quad+\cdots\\&=c(sI-A)^{-1}[I-\beta D(sI-A)^{-1}]^{-1}[I-\beta D(-A)^{-1}]^{-1}b\end{aligned}$$

where the last equality is obtained by writing the infinite series as a product of two power series. Using the identity

$$\begin{aligned}&(sI-A)^{-1}[I-\beta D(sI-A)^{-1}]^{-1}[I-\beta D(-A)^{-1}]^{-1}\\&\quad=[sI-(A+\beta D)]^{-1}[I-\beta D(A+\beta D)^{-1}]\end{aligned}$$

gives

$$L_\beta(s)=c[sI-(A+D\beta)]^{-1}[b-D(A+D\beta)^{-1}b\beta].$$

Obvious modifications for the polynomial-system cases complete the proof. □

Roughly speaking, this result shows that all bilinear realizations with invertible A of $\hat{H}(s_1, \cdots, s_\infty)$, of whatever dimension and in any coordinates, yield internal linearizations with identical input-output behavior. Of course, minimal bilinear realizations are of most interest, and in this regard it is tempting to conjecture that a minimal (span reachable and observable) bilinear realization would yield a minimal (reachable and observable) internal linearization, except possibly for isolated values of β. However, computation of the internal linearization for Example 3.3, where $A = 0$, shows that $u_\delta(t)$ does not even enter the linearized state equation. Other examples show that the conjecture is false even for the simplest case where the bilinear state equation is a degree-2 homogeneous system with invertible A.

Remark 3.7: For polynomial systems and minimal linear-analytic realizations in Crouch's form, results along the lines of Theorems 3.2 and 3.5 can be given [11]. Again, simple examples show that such state equations can have linearizations with identically zero input-output behavior for all β.

IV. Conclusions

The representation of bilinear-realizable nonlinear systems in terms of regular transfer functions provides a convenient framework for the study of linearization from an input-output perspective. The notion of an input-output linearization introduced herein appears to be a natural and potentially useful complement to the notion of a linearized state equation. In particular, it has been shown that the input-output perspective is helpful in explaining some mildly startling phenomena that can occur. Also, it appears to be helpful in providing further characterizations of the information about a bilinear system embodied in its linearization.

Application of these ideas to the calculation of the linearization of an interconnected system in terms of subsystem linearizations also has been reported [12].

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Global External Linearization of Nonlinear Systems Via Feedback

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Abstract—**This note presents necessary conditions and sufficient conditions for an affine nonlinear system to be globally feedback equivalent to a controllable linear system over an open subset V of $\mathbb{R}^n$. When V equals $\mathbb{R}^n$, necessary and sufficient conditions are obtained.**

I. Preliminary

We consider a nonlinear system of the form

$$\dot{x}=f(x)+\sum_{i=1}^{m} g_i(x)u_i \tag{1}$$

where $x \in M$, M is an open subset of $\mathbb{R}^n$. $f(x)$ and $g_i(x)$, $i = 1, \cdots, m$, are C^∞ vector fields on M.

The pioneer work of external linearization of an affine nonlinear system using feedback was developed by Brockett [1]. Jakubczyk and Respondek [2] presented necessary and sufficient conditions of local external linearization for the multiple input case. Su [3] and Hunt *et al.* [4] simplified such conditions. Recently, Hunt *et al.* [5] gave a sufficient condition for global external linearizability of a single input system. In this note we discuss necessary conditions and sufficient conditions for global external linearization of (1).

Let us first fix some notations. We take M (or N) to be a manifold, $C^\infty(M)$ the set of C^∞ functions on M, and $V(M)$ the set of C^∞ vector fields on M. $L_f g \triangleq [f, g]$ for $f, g \in V(M)$. We will denote the differential of a C^∞ mapping ϕ by ϕ_*. If $\phi: M \to N$ is a diffeomorphism, then $\phi_*: T_p(M) \to T_{\phi(p)}(N)$ is an isomorphism. $Sp\{X_1, X_2, \cdots, X_k\}$ is the free submodule of $V(M)$ over the ring $C^\infty(M)$ generated by X_1, X_2,

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$\cdots, X_k \in V(M)$. $\Phi_t^X(p)$ is the integral curve of $X \in V(M)$ with initial condition $\Phi_0^X(p) = p$.

II. General Result

Now we consider system (1). For convenience, we introduce a set of indexes and some notations. Let $l_1, l_2, \cdots, l_N$ be positive integers, such that $m = l_1 \geq l_2 \geq \cdots \geq l_N > 0$ and $\Sigma_{i=1}^N l_i = n$. Set $p_0 = 0$ and define $p_k \triangleq \Sigma_{i=1}^k l_i$, $k = 1, 2, \cdots, N$. Let $C_m^\infty(U)$ be the set of m-dimensional vectors with $C^\infty(U)$ entries; $Gl(m, C^\infty(U))$ be the set of $m \times m$ nonsingular matrices with $C^\infty(U)$ entries.

Definition 1: System (1) is said to be *globally externally linearizable*, if there exist a diffeomorphism $\phi: M \to U$, U open in R^n; $\alpha \in C_m^\infty(U)$; and $\beta \in Gl(m, C^\infty(U))$, and there exist integers $m = l_1 \geq l_2 \geq \cdots \geq l_N > 0$, such that

$$\begin{aligned}\phi_*(f) + (\phi_*(g_1), \cdots, \phi_*(g_m))\alpha \\ = (0, \cdots, 0 | y_1, \cdots, y_{l_2} | y_{p_1+1}, \cdots, y_{p_1+l_3} | \cdots | y_{p_{N-2}+1}, \\ \cdots, y_{p_{N-2}+l_N})^T,\end{aligned} \tag{2}$$

$$(\phi_*(g_1), \cdots, \phi_*(g_m))\beta = (I_m | 0)^T. \tag{3}$$

Definition 1 implies that after state-space coordinate change ϕ, input coordinate change β, and feedback α, the system (1) will have Brunovsky canonical form on U.

Now we state our result about system (1). Let $G_1 \triangleq Sp\{g_1, \cdots, g_m\}$, $G_{i+1} = G_i + L_f G_i$, $i = 1, 2, \cdots$, where $L_f G_i = \text{span}\{L_f v | v \in G_i\}$. We have the following theorem.

Theorem 1: If system (1) is globally externally linearizable, then the following holds.

i) There exist vector fields $\bar{g}_1, \cdots, \bar{g}_m \in G_1$ and integers $m = l_1 \geq l_2 \geq \cdots \geq l_N > 0$ with $\Sigma_{i=1}^N l_i = n$ such that the n vectors $C \triangleq \{\bar{g}_1, \cdots, \bar{g}_m, L_f\bar{g}_1, \cdots, L_f\bar{g}_{l_2}, \cdots, L_f^{N-1}\bar{g}_1, \cdots, L_f^{N-1}\bar{g}_{l_N}\}$ are linearly independent at each $x \in M$.

ii) $G_i = Sp\{\bar{g}_1, \cdots, \bar{g}_m, L_f\bar{g}_1, \cdots, L_f\bar{g}_{l_2}, \cdots, L_f^{i-1}\bar{g}_1, \cdots, L_f^{i-1}\bar{g}_{l_i}\}$, $i = 1, 2, \cdots, N$.

iii) Let X_i be the ith vector field in C. We define $D_i \triangleq Sp\{X_1, \cdots, X_i\}$, then D_i is involutive for $i = 1, 2, \cdots, n$.

iv) There exist $Z_i \in D_i$, $i = 1, 2, \cdots, n$, $x_0 \in M$, such that $Z_1, Z_2, \cdots, Z_n$ are linearly independent $\forall x \in M$, and the mapping $\psi: V \to M$, defined as

$$\psi(y_1, y_2, \cdots, y_n) = \Phi_{y_1}^{Z_1} \circ \Phi_{y_2}^{Z_2} \circ \cdots \circ \Phi_{y_n}^{Z^n}(x_0)$$

is bijective, where V is an open subset of $\mathbb{R}^n$.

Conditions i), ii), iii), and iv) are also sufficient provided in iv) V is also convex.

III. Proof of Theorem 1

First we prove the necessity. From (2) and (3) we have

$$(\tilde{g}_1, \cdots, \tilde{g}) \triangleq (\phi_*(g_1), \cdots, \phi_*(g_m))\beta = (I_m | 0)^T, \tag{4}$$

$$\begin{aligned}\tilde{f} \triangleq \phi_*(f) \\ = (\times, \cdots, \times | y_1, \cdots, y_{l_1} | y_{p_1+1}, \cdots, y_{p_1+l_3} | \cdots | y_{p_{N-2}+1}, \cdots, y_{p_{N-2}+l_N})^T\end{aligned} \tag{5}$$

where $\times$ denotes an arbitrary element. Now let

$$\tilde{C} \triangleq \{\tilde{g}_1, \cdots, \tilde{g}_m, L_{\tilde{f}}\tilde{g}_1, \cdots, L_{\tilde{f}}\tilde{g}_{l_2}, \cdots, L_{\tilde{f}}^{N-1}\tilde{g}_1, \cdots, L_{\tilde{f}}^{N-1}\tilde{g}_{l_N}\};$$

$$\tilde{G}_1 \triangleq Sp\{\tilde{g}_1, \cdots, \tilde{g}_m\};$$

$$\tilde{G}_{k+1} \triangleq \tilde{G}_k + L_{\tilde{f}}\tilde{G}_k, \qquad k = 1, 2, \cdots,$$

$\tilde{D}_i \triangleq Sp\{\tilde{X}_1, \cdots, \tilde{X}_i\}$, where $\tilde{X}_j$ is the jth element of $\tilde{C}$, $i = 1, \cdots, n$. According to (4), (5), it is obvious that $\tilde{G}_k, \tilde{C}, \tilde{D}_i$ satisfy i), ii), and iii) of Theorem 1, respectively. So when we choose $\bar{g}_i = \phi_*^{-1}(\tilde{g}_i)$, $i = 1, \cdots, m$, i), ii), and iii) are satisfied.

Note that the matrix $\tilde{C}$ is upper triangular and nonsingular. So if we choose $\tilde{Z}_i = (0, \cdots, 0, 1_{i\text{th}}, 0, \cdots, 0)^T$, $i = 1, \cdots, n$ and $x_0 = 0$, then $\tilde{Z}_1, \cdots, \tilde{Z}_n$ satisfy condition iv) with $\tilde{\psi}: V \to V$ being an identity mapping. Set $Z_i = \phi_*^{-1}(\tilde{Z}_i)$, $i = 1, \cdots, n$, then the corresponding $\psi: V \to M$ is $\phi^{-1} \circ \tilde{\psi}$. Thus, condition iv) is satisfied.

Before we prove sufficiency, we introduce several lemmas.

Lemma 1 [6]: If $X_1, \cdots, X_k \in V(M)$, $\Delta = Sp\{X_1, \cdots, X_k\}$ is involutive, then

$$(\Phi_t^{X_i})_*\Delta \subseteq \Delta, \qquad i = 1, \cdots, k.$$

Lemma 2: Assume $Z_1, Z_2, \cdots, Z_n$ are linearly independent vector fields $\forall x \in M$ and the distributions $\Delta_i \triangleq Sp(Z_1, \cdots, Z_i)$, $i = 1, \cdots, n$, are involutive. Let V be an open subset of $\mathbb{R}^n$. For a fixed $x_0 \in M$ if the mapping $\psi: V \to M$, defined by

$$\psi(y_1, \cdots, y_n) = \Phi_{y_1}^{Z_1} \circ \cdots \circ \Phi_{y_n}^{Z_n}(x_0)$$

is bijective, then ψ is a diffeomorphism.

Proof: To simplify the notation, let

$$\phi^{(i,j)} \triangleq \phi_{y_i}^{Z_i} \circ \phi_{y_{i+1}}^{Z_{i+1}} \circ \cdots \circ \phi_{y_j}^{Z_j},$$

$$\phi^{-(i,j)} \triangleq \phi_{-y_j}^{Z_j} \circ \phi_{-y_{j-1}}^{Z_{j-1}} \circ \cdots \phi_{-y_i}^{Z_i}, \qquad \text{where } 1 \leq i < j \leq n.$$

Since ψ is bijective, we have only to show that the Jacobian matrix of ψ,

$$J = \left(\frac{\partial \psi}{\partial y_1}, \frac{\partial \psi}{\partial y_2}, \cdots, \frac{\partial \psi}{\partial y_n}\right)$$

is nonsingular everywhere.

If M, N are two manifolds, $F: M \to N$ is a diffeomorphism, $X \in V(M)$, $p \in M$, then from [7] we can obtain

$$F \circ \phi_t^X(p) = \phi_t^{F_*(X)}(F(p)).$$

Differentiating both sides of the above equation we have

$$\frac{d}{dt} F \circ \phi_t^X(p) = F_* X(\phi_t^X(p)).$$

Applying the above result, we get

$$\frac{\partial \psi}{\partial y_i} = \phi_*^{(1,i-1)} Z_i(\phi^{(i,n)}(x_0)), \qquad i = 1, \cdots, n.$$

Now assume that there exists $(\lambda_1, \cdots, \lambda_n) \neq 0$, such that at $y \in V$

$$\sum_{i=1}^n \lambda_i \frac{\partial \psi}{\partial y_i} = \sum_{i=1}^n \lambda_i \phi_*^{(1,i-1)} Z_i(\phi^{(i,n)}(x_0)) = 0. \tag{6}$$

Let $k = \max\{i | \lambda_i \neq 0\}$, then (6) becomes

$$\begin{aligned}&\sum_{i=1}^k \lambda_i \phi_*^{(1,i-1)} Z_i(\phi^{(i,n)}(x_0)) \\ &= \phi_*^{(1,k-1)} \left\{ \sum_{i=1}^{k-1} \lambda_i \phi_*^{-(i,k-1)} Z_i(\phi^{(i,n)}(x_0)) \right. \\ &\qquad \left. + \lambda_k Z_k(\phi^{(k,n)}(x_0)) \right\} = 0.\end{aligned} \tag{7}$$

Since $Z_i \in Sp\{X_1, \cdots, X_i\}$ and $Z_1, \cdots, Z_n$ are linearly independent $\forall p \in M$, so $Sp\{Z_1, \cdots, Z_i\} = Sp\{X_1, \cdots, X_i\} = \Delta_i$ is involutive for $i = 1, \cdots, n$. Using Lemma 1, (7) contradicts the fact that $Z_1, \cdots, Z_k$ are linearly independent everywhere.

Lemma 3: Let V be a convex open subset of $\mathbb{R}^n$, $H: V \to \mathbb{R}^n$ be a C^∞ mapping. If the Jacobian matrix of H is an upper triangular nonsingular matrix everywhere, then $H: V \to I(V)$ is a diffeomorphism, where $I(V)$ is the image of H.

Proof: Using the mean value theorem, we can prove that H is a one-to-one mapping. The conclusion follows.

Note: By the rank theorem [7], $I(V)$ is open in $\mathbb{R}^n$.

Now we prove the sufficiency.

By assumptions iii) and iv) we know that D_i is involutive for $i = 1, \cdots, n$, $Z_1, \cdots, Z_n$ are linearly independent, and $\psi: V \to M$ is bijective. Thus, from Lemma 2 ψ is a diffeomorphism. Now we choose (ψ^{-1}, M) as a coordinate chart. Under this new coordinate we have [2][1]

$$\bar{g}_1 = \begin{bmatrix} \times \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \bar{g}_2 = \begin{bmatrix} \times \\ \times \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \cdots, \bar{g}_m = \begin{bmatrix} \times \\ \cdot \\ \times \\ 0 \\ \cdot \\ 0 \end{bmatrix} \Big\} m$$

It is useful to remember that the diagonal elements are nonzero because $\bar{g}_1, \cdots, \bar{g}_m$ are linearly independent.

From the definition of G_i, $L_f^k \bar{g}_i$ always belongs to G_{k+1}. If $L_f^k \bar{g}_i$ belongs to C, computation shows that

$$\begin{cases} \dfrac{\partial f_{p_i+k}}{\partial y_{p_{i-1}+k}} \neq 0, \\[2mm] \dfrac{\partial f_{p_i+l}}{\partial y_{p_{i-1}+k}} = 0, \; l>k, \; i=1, \cdots, N-1; \; k=1, \cdots, l_{i+1}. \end{cases} \tag{8}$$

If $L_f^k \bar{g}_i$ does not belong to C, then for $l_i > l_{i+1}$, we have

$$\frac{\partial f_{p_i+l}}{\partial y_{p_{i-1}+k}} = 0, \; i=1, \cdots, N-1; \; l_{i+1}<k\le l_i; \; l>l_{i+1}. \tag{9}$$

Now denote $y_k^0 \triangleq y_k$, $f_k^0 \triangleq f_k$, $k = 1, \cdots, n$. We define transformations R_j inductively as

$$R_j : y_l^j = \begin{cases} f^{j-1}_{p_{i+1}+k}, & l=p_i+k, \; i=0, \cdots, N-2; \; k=1, \cdots, l_{i+2}, \\ y_l^{j-1}, & \text{otherwise} \end{cases} \tag{10}$$

where $f^j \triangleq (R_j)_* f$.

Using R_j, the linearization procedure is a generalization of [2]. The main difference is to show that all R_j, $j = 1, \cdots, N-1$, are global diffeomorphisms from V to their images $I_j(V)$, $j = 1, \cdots, N-1$.

First we know R_j is well defined. We claim that $R_j: V \to I_j(V)$ is a diffeomorphism. Computing the Jacobian matrix of R_j, assuming (8) and (9) hold for all f^j, it is clear that J_{R_j} is an upper triangular matrix with the diagonal elements a_{ll}^j, $l = 1, \cdots, n$, as

$$a_{ll}^j = \begin{cases} \dfrac{\partial f^{j-1}_{p_{i+1}+k}}{\partial y_{p_i+k}}, & l=p_i+k, \; i=0, \cdots, N-2; \; k=1, \cdots, l_{i+2}, \\ 1, & \text{otherwise} \end{cases} \tag{11}$$

By (8), J_{R_j} is nonsingular everywhere. Using Lemma 3, $I_j(V)$ is an open subset of R^n and $R_j: V \to I_j(V)$ is a diffeomorphism.

What remains to be proven is that if the components of f^{j-1}, as functions of $y_1, \cdots, y_n$, satisfy (8) and (9), then the components of f^j, as functions of $y_1, \cdots, y_n$ too, also satisfy (8) and (9). This will be proved as follows. Using (10), we need only consider $f^j_{p_i+k}$, $i = 0, 1, \cdots, N-2$; $1 \le k \le l_{i+2}$. Since

$$f^j_{p_i+l} = \sum_{h=p_i+l}^{n} \frac{\partial f^{j-1}_{p_{i+1}+l}}{\partial y_h} f_h^{j-1},$$

[1] Note that in this new coordinate system we work with $\psi_*^{-1}(f)$, $\psi_*^{-1}(\bar{g}_i)$, $i = 1, \cdots, m$. For convenience, they are denoted by $f, g_1, \cdots, g_m$ again. But we should keep this fact in mind.

thus we have

$$\frac{\partial f^j_{p_i+k}}{\partial y_{p_{i-1}+k}} = \frac{\partial f^{j-1}_{p_{i+1}+k}}{\partial y_{p_i+k}} \cdot \frac{\partial f^{j-1}_{p_i+k}}{\partial y_{p_{i-1}+k}} \neq 0 \tag{12}$$

and

$$\frac{\partial f^j_{p_i+l}}{\partial y_{p_{i-1}+k}} = 0, \; l>k \text{ or } l_{i+1}<k\le l_i \text{ and } l>l_{i+1}.$$

Therefore, (8) and (9) are also true for f^j. This fact makes it possible to construct the diffeomorphism $R_j: V \to I_j(V)$ inductively.

Now we claim that

$$f^j_{p_i+k} = y^j_{p_{i-1}+k}, \qquad i=N-j, \cdots, N-1, \; k=1, \cdots, l_{i-1}. \tag{13}$$

Consider the diffeomorphism $R_j^{j-1}: I_{j-1}(V) \to I_j(V)$, $R_j^{j-1} \triangleq R_j \circ (R_{j-1})^{-1}$. Observe that (13) is true for $j = 1$. So we assume that (13) is true for $j-1$. Using (10) and (13) the Jacobian matrix of R_j^{j-1} is

$$J_{R_j^{j-1}} = \left[\begin{array}{c} \times \\ \hline 0 \,|\, I_{n-p_{N-j}} \end{array} \right].$$

It is easy to see that $f^j \triangleq (R_j)_* f = (R_j^{j-1})_* f^{j-1}$ satisfies (13).

Let $U \triangleq I_{N-1}(V)$, which is an open subset of $\mathbb{R}^n$. Define ϕ to be $R_{N-1} \circ \psi^{-1}$, then $\phi: M \to U$ is a diffeomorphism. Using (13) and[1] we have

$$\phi_*(f) = (h_1, \cdots, h_m | y_1, \cdots, \cdots, y_{l_2} | y_{p_1+1}, \cdots, y_{p_1+l_3} | \cdots | y_{p_{N-2}+1}, \cdots, y_{p_{N-2}+l_N})^T$$

where y_i is used to denote y_i^{N-1}, $i = 1, \cdots, n$; and $h_i \in C^\infty(U)$, $i = 1, \cdots, m$. Since the Jacobian matrix of R_{N-1} is upper triangular and nonsingular, we have

$$(\phi_*(\bar{g}_1), \cdots, \phi_*(\bar{g}_m)) = [W^T|0]^T, \text{ where } W \in Gl(m, C^\infty(U)).$$

Now since $Sp\{g_1, \cdots, g_m\} = Sp\{\bar{g}_1, \cdots, \bar{g}_m\} = G_1$, so we can find an $m \times m$ nonsingular matrix H pointwise, such that

$$(g_1, \cdots, g_m) = (\bar{g}_1, \cdots, \bar{g}_m)H$$

where

$$H = [(\bar{g}_1, \cdots, \bar{g}_m)^T(\bar{g}_1, \cdots, \bar{g}_m)]^{-1}(\bar{g}_1, \cdots, \bar{g}_m)^T \cdot (g_1, \cdots, g_m) \in Gl(m, C^\infty(M)).$$

Thus, we have

$$(\phi_*(g_1), \cdots, \phi_*(g_m)) = (\phi_*(\bar{g}_1), \cdots, \phi_*(\bar{g}_m)) \circ (H \circ \phi^{-1})$$

where $H \circ \phi^{-1} \in Gl(m, C^\infty(U))$.

Let

$$\beta \triangleq [W(H \circ \phi^{-1})]^{-1} \in Gl(m, C^\infty(U)),$$

$$\alpha = -\beta \cdot (h_1, \cdots, h_m)^T \in C_m^\infty(u),$$

then

$$(\phi_*(f) + (\phi_*(g_1), \cdots, \phi_*(g_m))\alpha, (\phi_*(g_1), \cdots, \phi_*(g_m))\beta)$$

has the Brunovsky canonical form.

Remark: Theorem 1, i) says that the set C spans an n-dimensional space at each x; ii) is the selection of Kronecker indexes; iii) means that the vectors in C behave like basis vectors of a linear vector space; and iv) ensures that at each point the integral curves of $Z_1, \cdots, Z_n$ form the skeleton of a coordinate system. Note that conditions i), ii), and iii) are the necessary and sufficient conditions for system (1) to be locally externally linearizable. Thus, these are equivalent to the conditions of Hunt *et al.* [4].

IV. Single Input Case

For single input case we need the following lemma which follows from [3].

Lemma 4: Let $f, g \in V(M)$, if $g, L_f g, \cdots, L_f^{n-1} g$ are linearly independent, $\forall p \in M$, and $\Delta_{n-1} \triangleq Sp\{g, L_f g, \cdots, L_f^{n-2} g\}$ is involutive, then $\Delta_i \triangleq Sp\{g, L_f g, \cdots, L_f^{i-1} g\}$ is involutive and dim $(\Delta_i) = i$ for $i = 1, 2, \cdots, n$.

Using Theorem 1 and Lemma 4, we have the following.

Theorem 2: If the single input system

$$\dot{x} = f(x) + g(x)u \tag{14}$$

is globally externally linearizable, then the following holds.

i) $X_i \triangleq L_f^{i-1} g$, $i = 1, \cdots, n$, are linearly independent, $\forall p \in M$.

ii) $\Delta_{n-1} \triangleq Sp\{X_1, \cdots, X_{n-1}\}$ is involutive.

iii) There exist $Z_i \in Sp\{g, L_f g, \cdots, L_f^{i-1} g\}$, $i = 1, \cdots, n$, which are linearly independent $\forall p \in M$, V open in $\mathbb{R}^n$ and $x_0 \in M$ such that the mapping $\psi: V \to M$, defined as $(y_1, y_2, \cdots, y_n) \to \phi_{y_1}^{Z_1} \circ \phi_{y_2}^{Z_2} \circ \cdots \circ \phi_{y_n}^{Z_n}(x_0)$, is bijective.

Conditions i), ii), and iii) are also sufficient provided in iii) V is also convex.

A direct proof of Theorem 2 can be found in [10].

V. Global R^n Linearizability

In practice, it is useful to have the linearized system defined over $\mathbb{R}^n$. In this case (1) is said to be globally externally R^n linearizable. For global external R^n linearizability we have the following theorem.

Theorem 3: System (1) is globally externally R^n linearizable if and only if i), ii), and iii) of Theorem 1 hold and in iv) of Theorem 1, $V = \mathbb{R}^n$ and there exists $\epsilon > 0$ such that

$$\left| \frac{\partial f_{p_{i+1}+k}}{\partial y_{p_i+k}} \right| \geq \epsilon, \qquad i = 0, \cdots, N-2, \quad k = 1, \cdots, l_{i+2} \tag{14}$$

where $(y_1, \cdots, y_n)$ is the coordinate of ψ^{-1}. $f_1, \cdots, f_n$ are components of f expressed under coordinate ψ^{-1}.

Proof: If we choose $Z_i = \phi_*^{-1}(0, \cdots, 0, 1_{ith}, 0, \cdots, 0)^T$, then we have $\psi = \phi^{-1}$ and under ψ^{-1}, f has form (5). Take $\epsilon = 1$, then (14) is true. Hence, the necessity follows.

For sufficiency, R^n is obviously convex. From (11) and (12) we can inductively prove that the diagonal elements of the Jacobian matrix of R_j satisfy

$$\min_l |a_{ll}^j| \geq \min (\epsilon^{2^{j-1}}, 1), \qquad j = 1, \cdots, N-1.$$

Thus, from [8], $R_j: R^n \to R^n$ is a diffeomorphism.

VI. Example

Consider the following system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\sin x_2 \cos x_2 \\ \sin^2 x_2 \end{pmatrix} + \begin{pmatrix} e^{-x_1} \sin x_2 \\ e^{-x_1} \cos x_2 \end{pmatrix} u. \tag{15}$$

Boothby [7] showed that (15) satisfies the local conditions i) and ii) of Theorem 2, but it is not globally externally $\mathbb{R}^2$ linearizable.

Let $D_1 = Sp(g)$, $D_2 = Sp(g, L_f g)$, then we can prove that no matter how to choose $Z_1 \in D_1$, $Z_2 \in D_2$, the mapping

$$\psi(t_1, t_2) \triangleq \phi_{t_1}^{Z_1} \circ \phi_{t_2}^{Z_2}(x_0)$$

will never be onto $\mathbb{R}^2$. So condition iii) of Theorem 2 fails.

The details of the above example and other interesting examples can be found in [10].

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On the Causal Factorization Problem

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Abstract—**In this note the formalism of the infinite zero module and of the infinite pole module is used to analyze the causal factorization problem. This allows us to express its solutions in terms of zeros and poles at infinity and to have a better understanding of both the problem and the techniques previously proposed by various authors to solve it. Moreover, we obtain a characterization, in terms of the existence of certain causal factorizations, of the transfer functions having no infinite zeros.**

Introduction

In recent years many concepts arising in the theory of linear systems have been considered from an algebraic point of view. This fact has produced a better insight into various problems such as, for instance, state and output feedback [8], [6], output injection [5], exact model matching [4], [3], inverse dynamical systems [17], [1], and others. Moreover, the algebraic approach has given the possibility of extending known techniques to more general situations such as those concerning systems with coefficients in a ring which represent families of parameter dependent systems, 2-D systems or systems with delays (see, in addition to some of the above-mentioned papers, [7], [10], and the references therein).

In this framework, B. Wyman and M. Sain introduced, in [17], the notion of (finite) zero module which gives an abstract and module theoretic description of the classical [14] finite zero structure of a transfer function. The significance of the zero module in connection with inverse systems and blocked signal transmission was pointed out in [17], [18] and the theory was extended to the case of systems with coefficients in a ring in [1].

More recently, the notions of infinite zero module and of infinite pole module, which describe the structure at ∞ of a transfer function in module theoretic terms, have been introduced in [2]. Using these algebraic tools, some results on the minimality of inverse systems and connections with the geometric theory can be established.

In this note, our aim is to analyze the causal factorization problem using the formalism of the infinite zero module and of the infinite pole module. This enables us to compare different technical solutions of the problem, given in [11] and in [6], with the natural conditions, restated in a precise and meaningful sense, which concern the infinite zeros and poles.

In our opinion, this approach results in a better understanding of both the problem and the techniques used in [11] and [6] to solve it. Moreover, it gives a clear example of the naturality and significance of the algebraic point of view in dealing with zeros and poles of transfer functions.

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