International Journal of Control
Vol. 81, No. 10, October 2008, 1590–1599

Stability and stabilisation of planar switched linear systems via LaSalle's invariance principle

Yanyan Yuan* and Daizhan Cheng

Institute of Systems Science, Chinese Academy of Sciences, Beijing 100190, P.R. China

(Received 23 January 2007; final version received 7 November 2007)

This paper investigates the problem of uniformly asymptotical stability (UAS) and stabilisation of planar switched linear systems using LaSalle's invariance principle of switched systems. First, we show that a common weak quadratic Lyapunov function (WQLF) is enough to assure the UAS of a switched linear system with stable modes. Then the necessary and sufficient conditions for the existence of common WQLF are obtained. Secondly, we consider the problem of uniformly asymptotical stabilisation (UASZ) of single-input planar switched linear systems. Necessary and sufficient conditions for the closed-loop system with proper feedback to share a common WQLF are presented. It is also proved that a common WQLF is enough to assure the UAS of the closed-loop system.

Keywords: switched linear systems; common weak Lyapunov function; LaSalle's invariance principle; uniformly asymptotical stablility

1. Introduction

The problem of stability and stabilisation is one of the most interesting and challenging topics for switched systems. It is mainly because switchings cause some unpredicted phenomena: switching among stable modes may result in unstable trajectories and vice versa. To assure the stability, common Lyapunov function has been studied extensively (Dayawansa and Martin 1999; Mancilla-Aguilar and Garcia 2000; Mancilla-Aguilar 2000).

It was shown in several literature (e.g., Mancilla-Aguilar and Garcia (2000), Mancilla-Aguilar (2000)) that if there is a common Lyapunov function for all switching modes, then the switched system is globally uniformly asymptotically stable (GUAS) with respect to a compact set of switching modes, and conversely, if a switched system is GUAS, then there must exist a common Lyapunov function. Hence, seeking a common Lyapunov function becomes a most commonly used tool for stability and stabilisation analysis of switched systems. However, finding a common Lyapunov function is a hard topic, even if common quadratic Lyapunov function for switched linear systems has been investigated by many authors (Mason, Boscain and Chitour 2006; Shorten and Narendra 1997, 2000). A general stability criterion for planar switched linear systems was presented in Holcman and Margaliot (2003).

In Cheng (2004), a necessary and sufficient condition is given for stabilising the planar switched control systems using common quadratic Lyapunov function. This work is a follow-up of Cheng, Guo and Huang (2003) and Cheng (2004), and some similar approaches have been implemented. A new concept, namely, common weak quadratic Lyapunov function (WQLF), is proposed in this paper. It is shown in this paper that common WQLF generalises the known results and it provides a more powerful tool in stability analysis and stabiliser design.

Consider a switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(t) \in \mathbb{R}^n, \tag{1}$$

and a switched linear control system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m, \tag{2}$$

where the switching signal $\sigma(t) : [0, \infty) \to \Lambda = \{1, 2, \ldots, N\}$ is a right-continuous piecewise constant mapping.

We give some definitions for stability and stabilisation of system (1) and (2) respectively.

*Corresponding author. Email: yanyanyuan0311@163.com

ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207170701793814
http://www.informaworld.com

Definition 1: System (1) is said to be stable, if for each switching signal $\sigma(t)$ and for any $\varepsilon > 0$, there is a $\delta(\sigma, \varepsilon) > 0$ such that when $\|x_\sigma(0)\| \le \delta(\sigma, \varepsilon)$ the solution $x_\sigma(t)$ of (1) satisfies $\|x_\sigma(t)\| \le \varepsilon$, $t \in [0, +\infty)$.

Moreover, if $\delta(\sigma, \varepsilon) = \delta(\varepsilon)$ is independent of σ, system (1) is said to be uniformly stable with respect to σ.

Definition 2: System (1) is said to be asymptotically stable, if it is stable, and for each switching signal $\sigma(t)$, there exists an $M(\sigma) > 0$ such that for any $\varepsilon > 0$, there is a $T(\sigma, \varepsilon) > 0$, as long as $\|x_\sigma(0)\| < M(\sigma)$, we have $\|x_\sigma(t)\| \le \varepsilon, \quad \forall t > T(\sigma, \varepsilon)$.

Moreover, if $T(\sigma, \varepsilon) = T(\varepsilon)$ is independent of σ, system (1) is said to be uniformly asymptotically stable (UAS) with respect to σ.

Definition 3: System (2) is said to be uniformly asymptotically stabilisable, if there is a (could be switch-depending) feedback control such that the closed-loop system of (2) is uniformly asymptotically stable.

For switched linear systems it is natural to search a common quadratic Lyapunov function for all modes.

Definition 4: A positive definite quadratic form $x^T P x$ (or briefly, P) is called a common quadratic Lyapunov function (*CQLF*) for system (1) (or set $\{A_\lambda | \lambda \in \Lambda\}$) if

$$PA_\lambda + A_\lambda^T P = -Q_\lambda < 0, \quad \forall \lambda \in \Lambda. \tag{3}$$

$x^T P x$ (or briefly, P) is called a common weak quadratic Lyapunov function (common WQLF) if in (3) $Q_\lambda \ge 0$, $\forall \lambda \in \Lambda$. A common WQLF is called a diagonal common WQLF if P is diagonal.

For switched linear control systems, we may seek feedbacks $u_\lambda = K_\lambda x, \lambda \in \Lambda$ such that

$$\left\{ \tilde{A}_\lambda = A_\lambda + B_\lambda K_\lambda \middle| \lambda \in \Lambda \right\}$$

share a CQLF (Cheng 2004). The advantage for this approach is that the quadratic form is easily computable, though the existence of CQLF is not necessary for stability (Dayawansa and Martin 1999).

Consider system (1) or (2). A switched system is said to have a non–vanishing dwell time, if there exists a positive time period τ_0, such that the switching moments $\{\tau_k | k = 1, 2, \ldots\}$ satisfy

$$\inf_k (\tau_{k+1} - \tau_k) \ge \tau_0.$$

Through this paper we assume that

A1. Admissible switching signals have a dwell time $\tau_0 > 0$.

One of the basic motivations of this paper is from the following LaSalle's invariance principle.

Theorem 1 (Hespanha 2004): *Suppose that there exists a set $\{P_\lambda | \lambda \in \Lambda\}$ of symmetric positive definite matrices such that at each switching moment we have*

$$x^T(t) P_{\sigma(t)} x(t) \le x^T(t) P_{\sigma(t^-)} x(t), \tag{4}$$

and

$$P_\lambda A_\lambda + A_\lambda^T P_\lambda \le -C_\lambda^T C_\lambda, \quad \forall \lambda \in \Lambda \tag{5}$$

for an appropriately defined compact set of matrices $\{C_\lambda | \lambda \in \Lambda\}$. Then system (1) *is stable. Moreover, if each pair (C_λ, A_λ) is observable, then* (1) *is uniformly asymptotically stable.*

Note that if there is only one P (equivalently, $P_\lambda = P$, $\forall \lambda \in \Lambda$), then (3) is trivially true. Hence for system (1) or system (2), instead of searching a CQLF, we may find a common WQLF, and to see when it is enough to assure the UAS of switched systems.

For system (1) we prove that as long as all the switching modes are stable, common WQLF is enough to assure the UAS of the system. Then we give necessary and sufficient conditions for the existence of common WQLF. For system (2) we prove that in planar case common WQLF is also enough to assure the UAS. Then we also provide necessary and sufficient conditions for the existence of controls which assure the existence of common WQLF for the corresponding closed-loop system. The design of stabilisers is also presented.

This paper is organised as follows: § 2 shows that common WQLF is enough to assure the UAS; § 3 gives necessary and sufficient conditions for the existence of common WQLF. In § 4, we consider the uniformly asymptotical stabilisation (UASZ) of switched planar linear systems (2) and show that common WQLF is also enough to assure UASZ. Then the necessary and sufficient conditions for the existence of linear state feedback controls which assure the existence of common WQLF of the corresponding closed-loop system are also presented. The last section contains some concluding remarks.

2. UAS via common WQLF

In this paper, we only consider a finite set of switching modes. That is,

A2. The set of switching modes is finite. Precisely, $\Lambda = \{1, 2, \ldots, N\}$.

In this section we would like to show that a common WQLF is enough to assure the UAS of system (1). We need Lemma 1.

Lemma 1 (Harry, Anton and Malo 2001): *Consider a linear system*

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n. \tag{6}$$

Assume that C is a $p \times n$ matrix, and observe the following statements:

(i) *A is a stable matrix*;
(ii) *(C, A) is observable*;
(iii) *Equation $A^T P + PA = -C^T C$ has a positive definite solution P.*

Then any two of these statements imply the third.

Now observe system (1), if we allow the switching being arbitrary, each A_λ should be stable. So we assume

A3. A_λ, $\lambda \in \Lambda$ are stable.

Theorem 2: *Consider system* (1). *Assume A*3 *holds. Then a common WQLF is enough to assure the UAS.*

Proof: For each mode the statements (i) and (iii) of Lemma 1 are satisfied. So (ii) is satisfied, too. Using Theorem 1, the conclusion follows. □

From Theorem 2, the problem of UAS is converted to the problem of finding a common WQLF.

3. Common WQLF for switched planar linear systems

In this section, we investigate the common WQLF for system (1). Before presenting our main results, we give the following two lemmas. The proofs are the same as those for the corresponding lemmas of CQLF in Cheng et al. (2003).

Lemma 2: *Assume a set of matrices $\{A_\lambda | \lambda \in \Lambda\}$ are stable, i.e. $Re\sigma(A_\lambda) < 0$, (where $\sigma(A)$ is the set of eigenvalues of A) and there exists a common WQLF, then there exists an orthogonal matrix T such that $\{\tilde{A}_\lambda = T^T A_\lambda T | \lambda \in \Lambda\}$ has a common diagonal WQLF.*

Lemma 3: *Assume a matrix A has a diagonal WQLF, then its diagonal elements are all non-positive*, *i.e.*, $a_{ii} \le 0$, $i = 1, \ldots, n$.

According to Lemma 2, instead of searching a common WQLF we can search a diagonal common WQLF under certain orthogonal transformation on $\{A_\lambda | \lambda \in \Lambda\}$.

In the following, we will consider the planar switched linear systems. That is, assume in system (1), $n = 2$.

Note that an orthogonal transformation $T \in SO(2, R)$ can be expressed as

$$T_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad 0 \le t < 2\pi. \tag{7}$$

Consider a stable matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

According to Lemma 3, we first consider when

$$A(t) = {T_t}^T A T_t, \quad 0 \le t < 2\pi,$$

has non–positive diagonal elements.

For notational ease, set $S = \sin t, C = \cos t$, $S_2 = \sin(2t), C_2 = \cos(2t)$. Consequently,

$$A(t) = \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} C & -S \\ S & C \end{pmatrix}$$
$$= \begin{pmatrix} \alpha C^2 + (\beta+\gamma)SC + \delta S^2 & \beta C^2 + (\delta-\alpha)SC - \gamma S^2 \\ \gamma C^2 + (\delta-\alpha)SC - \beta S^2 & \delta C^2 - (\beta+\gamma)SC + \alpha S^2 \end{pmatrix},$$

Set

$$a = \frac{\alpha+\delta}{2}, \quad b = \frac{\alpha-\delta}{2}, \quad c = \frac{\beta+\gamma}{2}, \quad d = \frac{\beta-\gamma}{2},$$

then

$$A(t) = \begin{pmatrix} a + bC_2 + cS_2 & d + cC_2 - bS_2 \\ -d + cC_2 - bS_2 & a - bC_2 - cS_2 \end{pmatrix}. \tag{8}$$

Remark 1: From (8) one sees that we do not need to consider the whole $0 \le t < 2\pi$. It is enough to consider the problem only for $0 \le t < \pi$.

Let $r = \sqrt{b^2 + c^2}$. Using Lemma 3, then similar to Cheng et al. (2003), we have the following results:

Proposition 1: *Given a stable matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. When $r \le -a$, the diagonal elements of*

$$A(t) = {T_t}^T A T_t, \quad 0 \le t < \pi$$

are always non–positive. When $r \ge -a$, the diagonal elements are non-positive iff $t \in \Theta$, where $\Theta = \{t | 0 \le t < \pi,\ r|\sin(2t + \mu)| \le -a\}$, and $\mu \in [0, 2\pi)$ is determined by

$$\cos(\mu) = \frac{c}{r}, \quad \sin(\mu) = \frac{b}{r}.$$

Theorem 3: *For each $t \in \Theta$, there exists a non–empty interval*

$$I_t = [L(t), U(t)] \subset (0, +\infty),$$

such that $P = \mathrm{diag}(1, x)$ is a diagonal WQLF of $A(t)$, iff, $x \in I_t$, where

$$L(t) = \begin{cases} 1 + 2\left(\frac{a}{d}\right)^2 - \left(\frac{2|a|}{|d|}\right)\sqrt{\left(\frac{a}{d}\right)^2 + 1}, & r = 0, \\ \frac{(RC+d)^2}{-2F}, & r > 0,\ RC = d, \\ \frac{-F - \sqrt{F^2 - ((RC)^2 - d^2)^2}}{(RC-d)^2}, & r > 0,\ RC \neq d; \end{cases} \tag{9}$$

$$U(t) = \begin{cases} 1 + 2\left(\frac{a}{d}\right)^2 + \left(\frac{2|a|}{|d|}\right)\sqrt{\left(\frac{a}{d}\right)^2 + 1}, & r = 0, \\ +\infty, & r > 0,\ RC = d, \\ \frac{-F + \sqrt{F^2 - ((RC)^2 - d^2)^2}}{(RC-d)^2}, \\ r > 0,\ RC \neq d, t \in \Theta. \end{cases} \tag{10}$$

$$RC = r\cos(2t + \mu), \quad RS = r\sin(2t + \mu),$$
$$F = (RC)^2 - d^2 + 2(RS)^2 - 2a^2.$$

Now for a given 2×2 stable matrix A, the set of WQLFs can be described as

$$\left\{P = pT_t\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}T_t^T | p > 0,\ t \in \Theta,\ L(t) \leq x \leq U(t)\right\}.$$

In the following, we use $(t, x) \in [0, \pi) \times (0, +\infty)$ to represent the set of candidates of WQLFs. That is

$$P(t, x) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Similar to CQLF Cheng et al. (2003), we have

Proposition 2: *If $P(t, x)$ is a feasible common WQLF with $t < \pi/2$, then $P(t + \pi/2, 1/x)$ is also a feasible common WQLF. Conversely, if $P(t, x)$ is a feasible common WQLF with $t \geq \pi/2$, then $P(t - \pi/2, 1/x)$ is also a feasible common WQLF.*

This proposition tells us that to search a common WQLF we have only to search it over $[0, \pi/2)$.

Now consider the finite set $\{A_\lambda | \lambda \in \Lambda\}$, then we can construct the feasible set of t for each matrix, as Θ_λ. And the functions $L_\lambda(t), U_\lambda(t)$ are defined by (9) and (10). Then the common feasible set is

$$\Theta = \cap_{\lambda=1}^N \Theta_\lambda \subset \left[0, \frac{\pi}{2}\right).$$

We also set

$$L(t) = \max_{\lambda \in \Lambda} L_\lambda(t), \quad U(t) = \min_{\lambda \in \Lambda} U_\lambda(t), \quad t \in \Theta.$$

Summarising the above arguments, we have the following result.

Theorem 4: *Consider system (1), $\{A_\lambda | \lambda \in \Lambda\}$ share a common WQLF, iff there exists a $t \in \Theta$, such that $L(t) \leq U(t)$. Moreover, as long as a common WQLF exists, system (1) is uniformly asymptotically stable.* □

Note that if the conditions in Theorem 4 are satisfied, then a common WQLF can be constructed as

$$P = P(t, x), \quad L(t) \leq x \leq U(t), \quad t \in \Theta.$$

Example 1: Consider the following switched system

$$\dot{x} = A_{\sigma(t)}x, \tag{11}$$

where $\sigma(t)\colon [0, +\infty) \to \{1, 2\}$, $x \in \mathbb{R}^2$ with switching modes as

$$A_1 = \begin{pmatrix} -4 & 6 \\ -2 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -4 \\ 3 & -3 \end{pmatrix}.$$

The parameters of A_1 are calculated as

$$a_1 = -1, \quad b_1 = -3, \quad c_1 = 2, \quad d_1 = 4, \quad r_1 = \sqrt{13},$$
$$\mu_1 = 5.3004.$$

It follows that $\Theta_1 = (0.3509,\ 0.6319)$.

From (9) and (10), we have

$$L_1(t) = \frac{5 - 13(\sin(2t+\mu_1))^2 - 4\sqrt{1 - 13(\sin(2t+\mu_1))^2}}{(\sqrt{13}\cos(2t+\mu_1) - 4)^2},$$

$$U_1(t) = \frac{5 - 13(\sin(2t+\mu_1))^2 + 4\sqrt{1 - 13(\sin(2t+\mu_1))^2}}{(\sqrt{13}\cos(2t+\mu_1) - 4)^2},$$

$$t \in \Theta_1,$$

Similarly, the parameters of A_2 are

$$a_2 = -1, \quad b_2 = 2, \quad c_2 = -\frac{1}{2}, \quad d_2 = -\frac{7}{2},$$
$$r_2 = \frac{\sqrt{17}}{2}, \quad \mu_2 = 1.8158,$$

and $\Theta_2 = (0.4097,\ 0.9161)$. Therefore

$$\Theta = \Theta_1 \cap \Theta_2 = (0.4097, 0.6319).$$

Using (9) and (10) again, we have

$$L_2(t) = \frac{40 - 17(\sin(2t+\mu_2))^2 - 12\sqrt{4 - 17(\sin(2t+\mu_2))^2}}{(\sqrt{17}\cos(2t+\mu_2)+7)^2},$$
$$U_2(t) = \frac{40 - 17(\sin(2t+\mu_2))^2 + 12\sqrt{4 - 17(\sin(2t+\mu_2))^2}}{(\sqrt{17}\cos(2t+\mu_2)+7)^2},$$
$$t \in \Theta_2,$$

Let

$$L(t) = \max\{L_1(t), L_2(t)\},\ U(t) = \min\{U_1(t), U_2(t)\}, \quad t \in \Theta.$$

The portraits of $L_1 - U_1$, $L_2 - U_2$ and $L - U$ are described in Figure 1.

From Figure 1, it is easy to find out a common WQLF $(t, x) = (0.5513,\ 6.8209)$. Then we have

$$P = \begin{pmatrix} 0.9989 & -0.9989 \\ -0.9989 & 2.0092 \end{pmatrix}.$$

It is easy to verify that

$$PA_1 + A_1^T P = \begin{pmatrix} -3.9951 & 3.9727 \\ 3.9727 & -3.9505 \end{pmatrix} \leq 0,$$
$$PA_2 + A_2^T P = \begin{pmatrix} -3.9959 & 4.0300 \\ 4.0030 & -4.0636 \end{pmatrix} \leq 0.$$

Its dual common WQLF is $(t, x) = (2.1221,\ 0.1466)$, which yields

$$\bar{P} = \begin{pmatrix} 0.9899 & -0.9900 \\ -0.9900 & 1.9912 \end{pmatrix}.$$

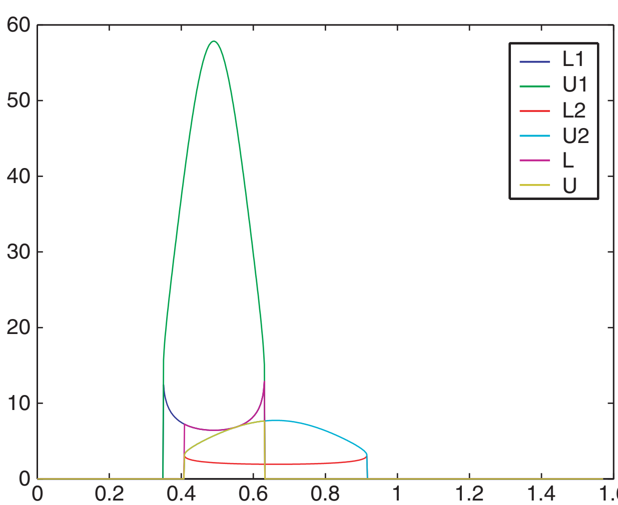

Figure 1. The portrait of $L_1 - U_1$, $L_2 - U_2$ and $L - U$.

Moreover, we have

$$\bar{P}A_1 + A_1^T \bar{P} = \begin{pmatrix} -3.9602 & 3.9372 \\ 3.9372 & -3.9153 \end{pmatrix} \leq 0,$$
$$\bar{P}A_2 + A_2^T \bar{P} = \begin{pmatrix} -3.9602 & 3.9940 \\ 3.9940 & -4.0273 \end{pmatrix} \leq 0.$$

So both P and $\bar{P}$ are common WQLFs of A_1 and A_2.

According to Theorem 4, system (11) is uniformly asymptotically stable.

Remark 2: In fact, the conclusion in § 3 is available for n dimensional switched systems. So searching a common WQLF is essential for the UAS. Similar to what we have done in this section, the algorithm developed in Cheng et al. (2003) can also be extended to search a common WQLF for higher dimensional case.

4. Stabilisation of switched planar linear systems

In this section, we consider the problem of the UASZ of planar switched systems (2) by linear feedback. First, we give a definition.

Definition 5: Consider switched system (2). The weak quadratic stabilisation problem is: finding feedback controls $u_\lambda = K_\lambda x$, $\lambda = 1, 2, \ldots, N$, and a positive definite matrix P, satisfying $P\tilde{A}_\lambda + \tilde{A}_\lambda^T P = -C_\lambda^T C_\lambda \leq 0$, where $\tilde{A}_\lambda = A_\lambda + B_\lambda K_\lambda$, C_λ, $\lambda = 1, 2, \ldots, N$ are appropriate matrices and (C_λ, A_λ) are observable.

Next, we give some lemmas for providing our main results.

Lemma 4 (Cheng (2004) modified a little): *Let (A, b) be a single-input planar system. Then there exists a unique state transformation matrix T, which converts the system into the Brunovsky canonical form as*

$$T^{-1}AT = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix}, \quad T^{-1}b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Moreover, the parameters a_i, $i = 1, 2$ satisfy

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (b, Ab)^{-1} A^2 b,$$

and the unique state transformation matrix $T = (T_1, T_2)$ can be determined as follows:

$$T_2 = b, \quad T_1 = AT_2 - a_2 b.$$

Lemma 5: *Given a positive definite matrix*

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0.$$

There exists a feedback $u = Kx = (k_1, k_2)x$, such that the closed-loop system of the canonical single-input planar system

$$\tilde{A} = A + bK$$

has M as its weak quadratic Lyapunov function, iff $m_2 \geq 0$.

Proof: Denote

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(k_1, k_2) := \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}.$$

Without loss of generality, we assume $m_1 = 1$. Then M is a weak quadratic Lyapunov function of $\tilde{A}$, iff

$$\tilde{A}^T M + M\tilde{A} = \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 1 & m_2 \\ m_2 & m_3 \end{pmatrix} + \begin{pmatrix} 1 & m_2 \\ m_2 & m_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \leq 0,$$

which leads to

$$Q := \tilde{A}^T M + M\tilde{A} = \begin{pmatrix} 2\alpha m_2 & 1 + \alpha m_3 + \beta m_2 \\ 1 + \alpha m_3 + \beta m_2 & 2(m_2 + \beta m_3) \end{pmatrix} \leq 0.$$

Denote $D(\alpha, \beta) := \det Q$. Now for Q to be negative semi-definite, it is necessary and sufficient that

$$\alpha m_2 \leq 0, \quad D(\alpha, \beta) \geq 0, m_2 + \beta m_3 \leq 0.$$

Note that $\beta \leq 0$ is a necessary condition for the closed-loop system to be stable. Meanwhile, $\alpha m_2 \leq 0$ is also necessary. Then we have

Case 1: $m_2 < 0$, $\alpha > 0$ and $\beta < 0$. After a simple computation, we have

$$D(\alpha, \beta) = -(\beta m_2 - \alpha m_3)^2 - 2(\beta m_2 + \alpha m_3) + 4\alpha m_2^2 - 1.$$

Setting

$$\frac{\partial D(\alpha, \beta)}{\partial \beta} = -2m_2(\beta m_2 - \alpha m_3) - 2m_2 = 0,$$

we have

$$\beta m_2 = \alpha m_3 - 1.$$

So the maximum of $D(\alpha, \beta)$ is

$$D_{\max}(\alpha, \beta) = -1 - 2(2\alpha m_3 - 1) + 4\alpha m_2^2 - 1 = -4\alpha(m_3 - m_2^2) = -4\alpha det(M) < 0.$$

It is obvious that $D(\alpha, \beta) \geq 0$ has no solution.

Case 2: $m_2 < 0, \alpha > 0$ and $\beta = 0$. Then we have

$$D(\alpha, \beta) = -(\alpha m_3)^2 - 2\alpha m_3 + 4\alpha m_2^2 - 1 = -(\alpha m_3 + 1)^2 + 4\alpha m_2^2.$$

Let

$$\frac{\partial D(\alpha, \beta)}{\partial \alpha} = -2\alpha m_3^2 + 4m_2^2 - 2m_3 = 0.$$

Then

$$\alpha = \frac{2m_2^2 - m_3}{m_3^2}.$$

So the maximum of $D(\alpha, \beta)$ is

$$D_{\max}(\alpha, \beta) = -\left(\frac{2m_2^2 - m_3}{m_3^2} + 1\right)^2 + 4m_2^2\frac{2m_2^2 - m_3}{m_3^2} = \frac{4m_2^2(m_2^2 - m_3)}{m_3^2} < 0.$$

Therefore, $D(\alpha, \beta) \geq 0$ has no solution.

Case 3: $m_2 < 0, \alpha = 0$ and $\beta < 0$. In this case, we have

$$D(\alpha, \beta) = -(\beta m_2 + 1)^2 < 0.$$

Case 4: $m_2 < 0, \alpha = 0$ and $\beta = 0$. Then we have $D(\alpha, \beta) = -1 < 0$. In case 3 and case 4, $D(\alpha, \beta) \geq 0$ does not have solution either. Hence $m_2 \geq 0$ is necessary for the matrix Q to be negative semi-definite, which proves the necessity.

Next, we prove the sufficiency.

Case 1: $m_2 > 0$. In this case, we can choose any $\alpha < 0$ and $\beta = ((\alpha m_3 - 1)/m_2)$. Then we have

$$D(\alpha, \beta) = -4\alpha(m_3 - m_2^2) > 0,$$

$$m_2 + \beta m_3 = m_2 + \frac{\alpha m_3 - 1}{m_2}m_3 = \frac{1}{m_2}[m_2^2 - m_3 + \alpha m_3^2] < 0.$$

Hence $Q < 0$.

Case 2: $m_2 = 0$. Then

$$Q = \begin{pmatrix} 0 & 1 + \alpha m_3 \\ 1 + \alpha m_3 & 2\beta m_3 \end{pmatrix}.$$

Setting $\alpha = -(1/m_3)$ and $\beta = -(1/m_3)$, a straightforward computation shows

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \leq 0.$$

The sufficiency is proved. □

Definition 6: A matrix

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0$$

is said to be weakly canonical-friend (to the canonical controllable form), if $m_2 \geq 0$.

Lemma 6: *Given a non-singular matrix*

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}.$$

For a weakly canonical-friend $M > 0$, $T^T M T$ *is also weakly canonical-friend, iff the following quadratic inequality*

$$t_{21}t_{22}x^2 + (t_{12}t_{21} + t_{11}t_{22})x + t_{11}t_{12} \geq 0$$

has a non-negative solution $x \geq 0$.

Proof: The proof is similar to the proof for the canonical-friend case in Cheng (2004). □

Consider the single-input planar switched system (2). For each switching mode we denote the state transformation matrix, which converts it to the canonical form, by $C_i, i = 1, \ldots, N$. That is, let

$$z_i = C_i x, \quad i = 1, \ldots, N.$$

Then the ith mode $\dot{x} = A_i x + b_i u$, when expressed into z_i coordinates, is in the Brunovsky canonical form.

Set $T_i = C_1 C_{i+1}^{-1}, i = 1, \ldots, N-1$. T_i is the state transformation matrix from z_{i+1} to z_1, that is

$$z_1 = C_1 x = C_1 C_{i+1}^{-1} z_{i+1} = T_i z_{i+1}.$$

Next, we classify $T_i = (t^i_{j,k})$ into three categories:

$$S_p = \{i \in \Lambda | t^i_{21} t^i_{22} > 0\}; \quad S_n = \{i \in \Lambda | t^i_{21} t^i_{22} < 0\}; \quad S_z = \{i \in \Lambda | t^i_{21} t^i_{22} = 0\}.$$

Then $\Lambda = S_p \cup S_n \cup S_z$. Next, for $i \in S_z$, we define

$$c_i = t^i_{12} t^i_{21} + t^i_{11} t^i_{22}, \quad d_i = t^i_{11} t^i_{12}.$$

and a linear form as

$$L_i = c_i x + d_i, \quad i \in S_z.$$

For $i \in S_p \cup S_n$, we define

$$a_i = \frac{t^i_{11}}{t^i_{21}}, \quad b_i = \frac{t^i_{12}}{t^i_{22}},$$

and a quadratic form as

$$Q_i = x^2 + (a_i + b_i)x + a_i b_i.$$

According to Lemma 6, we may solve x from

$$Q_i(x) < 0, \quad i \in S_n; \quad Q_i(x) > 0, \quad i \in S_p; \quad L_i(x) > 0, \quad i \in S_z.$$

Proposition 3: *Let* (A, b) *be a canonical planar system, i.e.,*

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

and

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0$$

be weakly canonical-friend (*i.e.* $m_2 \geq 0$). *To make* $x^T M x$ *a weakly quadratic Lyapunov function of the closed-loop system, a feedback control can be chosen as*

$$u = kx = \begin{cases} \left(\alpha - a_1, \dfrac{\alpha m_3 - m_1}{m_2} - a_2\right)x, & m_2 > 0, \\ \left(-\dfrac{m_1}{m_3} - a_1, -\dfrac{m_1}{m_3} - a_2\right)x, & m_2 = 0, \end{cases}$$

where $\alpha < 0$ *can be any negative real number.*

Theorem 5: *The closed-loop system of* (2) *has a common WQLF, iff there exists a non–negative* x *satisfying*

$$\max_{i \in S_n} Q_i(x) < 0, \quad \min_{i \in S_p} Q_i(x) \geq \max_{i \in S_n} Q_i(x), \quad L_i(x) \geq 0, \quad i \in S_z. \tag{12}$$

Proof: Assume there is a WQLF in z_1 coordinates, which is expressed as

$$M_1 = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0.$$

Suppose $\tilde{M}_1 = T^T M_1 T$ and set

$$H(m_1, m_2, m_3) := \tilde{m}_{12} = t_{21}t_{22}m_3 + (m_{12}t_{21} + t_{11}t_{22})m_2 + t_{11}t_{12}m_1.$$

Without loss of generality, we assume $m_1 = 1$.Then according to Lemma 5, $m_2 \geq 0$. It is easy to see from the proof of Lemma 6 that M_1 is a common WQLF for the other modes, iff $H_i(1, m_2, m_3) \geq 0$, $i = 1, \ldots, N-1$, which leads to

$$\left.\begin{aligned} m_3 + (a_i + b_i)m_2 + a_i b_i \geq 0, \quad & i \in S_p; \\ m_3 + (a_i + b_i)m_2 + a_i b_i \leq 0, \quad & i \in S_n; \\ c_i m_2 + d_i \geq 0, \quad & i \in S_z. \end{aligned}\right\} \tag{13}$$

Since $m_3 > m_2^2$, we can rewrite the first two inequalities as

$$e + m_2 + (a_i + b_i)m_2 + a_i b_i \geq 0, \quad i \in S_p; \tag{14}$$

$$e + m_2 + (a_i + b_i)m_2 + a_i b_i \leq 0, \quad i \in S_n, \tag{15}$$

where $e > 0$. The necessity of (12) is obvious. As for sufficiency, assume there exists a non–negative solution x such that the inequalities in (12) hold. If $\min_{i \in S_p} Q_i(x) \geq 0$, then we can choose $m_2 = x$ and $m_3 = x^2 + \varepsilon$. As $\varepsilon > 0$ is small enough, (13) is satisfied.

If $w = \min_{i \in S_p} Q_i(x) < 0$, we can choose $m_2 = x$ and $m_3 = x^2 + 1/2(\min_{i \in S_p} Q_i(x) - \max_{i \in S_n} Q_i(x)) - w$.

It is easy to see that for such a choice (14) and (15) hold. Hence the matrix

$$M_1 = \begin{pmatrix} 1 & m_2 \\ m_2 & m_3 \end{pmatrix}$$

meets the requirement. □

Corollary 1: *The closed-loop system of* (2) *has a common WQLF, iff the following set of linear inequalities have a solution*:

$$\min\{-a_j, -b_j\} < x < \max\{-a_j, -b_j\}, \quad j \in S_n,$$

$$(a_i + b_i - a_j - b_j)x + a_i b_i - a_j b_j \geq 0, \quad i \in S_p, j \in S_n,$$

$$c_i x + d_i \geq 0, \quad i \in S_z,$$

$$x \geq 0.$$

Lemma 7: *Let $Q = -C^T C$, where C is an approriate matrix. Then $(C, \tilde{A})$ is observable if we choose the feedback control u as in* Proposition 3.

Proof: According to Lemma 5, we have

$$\tilde{A} = A + bK := \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix},$$

$$Q := \tilde{A}^T M + M\tilde{A} = \begin{pmatrix} 2\alpha m_2 & 1 + \alpha m_3 + \beta m_2 \\ 1 + \alpha m_3 + \beta m_2 & 2(m_2 + \beta m_3) \end{pmatrix}$$
$$= -C^T C \leq 0.$$

Case 1: $m_2 > 0$.

In this case, we can choose $\alpha < 0$ and $\beta = (\alpha m_3 - 1)/m_2$. After a simple computation, we have $Q < 0$. It is obvious that

$$\text{rank}\begin{pmatrix} C \\ C\tilde{A} \end{pmatrix} = 2,$$

i.e., $(C, \tilde{A})$ is observable.

Case 2: $m_2 = 0$.

We can choose $\alpha = -(1/m_3)$ and $\beta = -(1/m_3)$, then

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} = -\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} \end{pmatrix} = -C^T C \leq 0.$$

Thus,

$$\text{rank}\begin{pmatrix} C \\ C\tilde{A} \end{pmatrix} = \text{rank}\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} = \text{rank}\begin{pmatrix} 0 & 1 \\ -\frac{1}{m_3} & -\frac{1}{m_3} \end{pmatrix} = 2,$$

and therefore $(C, \tilde{A})$ is observable. □

Using Theorem 5 and Lemma 7, it is easy to prove the following theorem.

Theorem 6: *For system* (2), *the weak quadratic stabilisation problem is solvable, if the conditions in Theorem 5 are satisfied.*

Example 2: Consider the following switched system

$$\dot{x} = A_{\sigma(t)} x + b_{\sigma(t)} u_{\sigma(t)}, \tag{16}$$

where $\sigma(t)$: $[0 + \infty) \to \{1, 2, 3\}$, $x \in \mathbb{R}^2$ with switching modes as

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix};$$

$$A_3 = \begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Suppose z_1, z_2 and z_3 are canonical coordinates of modes 1, 2, 3 respectively. That is, under z_i, mode i has Brunovsky canonical form. Let $z_i = C_i x, i = 1, 2, 3$. Using Lemma 4, we can get C_i as

$$C_1 = I_2, \quad C_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \quad C_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then the state transformation matrices between z_1 and z_i, $i = 2, 3$, determined by $T_i = C_1 C_{i+1}^{-1}$, $i = 1, 2$, are obtained as

$$T_1 = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

After a simple computation, we have $1 \in S_p$ and $a_1 = -(1/2)$, $b_1 = (1/2)$; $2 \in S_z$ and $c_1 = -1$, $d_1 = 0$. Using Corollary 1, inequalities (14) and (15), we have $x = 0$, $e \geq (1/4)$. Then we can choose $x = 0$ and $e = 2$.

Setting $m_2 = x$, $m_3 = x^2 + e$, we have

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Now converting (A_i, b_i) to z_i, $i = 1, 2, 3$ coordinates, we have

$$\tilde{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\tilde{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\tilde{A}_3 = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad \tilde{b}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To get feedback control law, we need to convert M_1 into z_2 and z_3 frames as

$$M_2 = {T_1}^T M_1 T_1 = \begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Using Proposition 3, the feedback controls can be chosen as follows:

$$k_1 = \left(-\frac{3}{2}, -\frac{1}{2}\right), \quad k_2 = \left(-2, -\frac{18}{7}\right), \quad k_3 = \left(\frac{5}{2}, \frac{7}{2}\right).$$

Then in the original coordinates x we have

$$K_1 = \left(-\frac{3}{2}, -\frac{1}{2}\right), K_2 = \left(-\frac{2}{7}, -\frac{8}{7}\right), K_3 = \left(-\frac{5}{2}, \frac{7}{2}\right).$$

Getting M_1 back to the original coordinates x as

$$M_0 = {C_1}^T M_1 C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

we have

$$M_0(A_1 + b_1 K_1) + (A_1 + b_1 K_1)^T M_0 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \le 0,$$

$$M_0(A_2 + b_2 K_2) + (A_2 + b_2 K_2)^T M_0 = \begin{pmatrix} -\frac{18}{7} & -\frac{16}{7} \\ -\frac{16}{7} & -\frac{36}{7} \end{pmatrix} < 0,$$

$$M_0(A_3 + b_3 K_3) + (A_3 + b_3 K_3)^T M_0 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \le 0.$$

Consequently, there is a common WQLF of the three modes. According to Lemma 7, the observability of every mode is assured.

According to LaSalle's invariance principle of switched systems in Theorem 1, system (16) is uniformly asymptotically stabilised by the linear state feedbacks.

5. Conclusion

In this paper, the uniformly asymptotical stability and stabilisation of planar switched systems were investigated via common WQLF. First, it was proved that for a switched linear system a common WQLF is enough to assure the UAS. Then the existence of a common WQLF for a set of stable matrices was studied, and necessary and sufficient conditions were obtained. Secondly, we consider the problem of UASZ of single-input planar switched linear systems. A necessary and sufficient condition for system with proper linear feedbacks to have a common WQLF was obtained. It was also shown that the existence of common WQLF assures the UASZ. Some examples were presented to illustrate the results.

The first part of the work can be extended to higher dimensional systems. To extend the second part of the work to higher dimensional case is a difficult work, which is left for further study. Similar to Xie et al. (2004), the second part of the work could be extended to discrete-time case.

Acknowledgements

The authors would like to thank the anonymous reviewers for their constructive and insightful comments for further improving the quality of this work. This work was supported by NNSF of China under Grants 60674022 and 60736022.

References

Cheng, D.Z., Guo, L., and Huang, J. (2003), 'On Quadratic Lyapunov Functions,' *IEEE Transactions on Automatic Control*, 48, 885–890.

Cheng, D.Z. (2004), 'Stabilisation of Planar Switched Systems,' *Systems Control Letter*, 51, 79–88.

Dayawansa, W.P., and Martin, C.F. (1999), 'A Converse Lyapunov Theorem for a Class of Dynamical Systems Which Undergo Switching,' *IEEE Transactions on Automatic Control*, 44, 751–760.

Harry, L.T., Anton, A.S., and Malo, H. (2001), *Control Theory for Linear Systems*, London: Springer-Verlag.

Hespanha, J.P. (2004), 'Uniform Stability of Switched Linear Systems: Extensions of LaSalle's Invarivance Principle,' *IEEE Transactions on Automatic Control*, 49, 470–482.

Holcman, D., and Margaliot, M. (2003), 'Stability Analysis of Second-Order Switched Homogeneous Systems,' *SIAM Journal on Control and Optimization*, 45, 1609–1625.

Mancilla-Aguilar, J.L. (2000), 'A Condition for the Stability of Switched Non–Linear Systems,' *IEEE Transactions on Automatic Control*, 45, 2077–2079.

Mancilla-Aguilar, J.L., and Garcia, R.A. (2000), 'A Converse Lyapunov Theorem for Non–Linear Switched Systems,' *Systems Control Letter*, 41, 67–71.

Mason, P., Boscain, U., and Chitour, Y. (2006), 'Common Polynomial Lyapunov Functions for Linear Switched Systems,' *SIAM Journal on Control and Optimization*, 45, 226–245.

Shorten, R.N., and Narendra, K.S. (1997), 'A Sufficient Condition for the Existence of a Common Lyapunov Function for Two Second-Order Linear Systems,' in *Proceedings of the 36th Conference on Decision and Control*, San Diego, CA, pp. 3521–3522.

Shorten, R.N., and Narendra, K.S. (2000), 'Necessary and Sufficient Condition for the Existence of a Common Quadratic Lyapunov Function for M-Stable Linear Second Order Systems,' in *Proceedings of 2000ACC*, Chicago, IL, pp. 359–363.

Xie, D., Xie, G., and Wang, L. (2004), 'Complete Characterization of Quadratic Lyapunov Function for Planar Discrete Systems,' *Communication in Nonlinear Science and Numerical Simulation*, 9, 405–416.