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RESEARCH ARTICLE

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Solving logic equation via matrix expression

Abstract A new matrix product, called semi-tensor product of matrices, is introduced. Using this, an algebraic expression of logic is proposed, where a logical variability is expressed as a vector, a logic function is expressed as a matrix and the function values are obtained by the product of matrix with its arguments' vectors. Under this framework, the problem of solving logic equations is investigated. For a static logic equation, we convert it into a set of linear algebraic equations. Then the solution becomes obvious. Some examples are presented to show that it is useful for logic infection.

Keywords logic equation, semi-tensor product, structure matrix, logic inference

1 Introduction

Logic plays an important role in many control problems. For instance, for discrete event systems, to check whether a min-max expression is inseparable, a logic equation has to be considered [1]. Multi-valued logic and fuzzy logic are the foundation of fuzzy control [2]. Logic-based control has been widely used in flight control [3,4]. Recently, inspired by the Human Genome Project, a new view of biology, called the systems approach, has emerged. We refer to Refs. [5,6] for a general introduction to system biology. The Boolean network, introduced first by Kauffman [7], and then developed by many other researches, becomes a powerful tool in describing, analyzing, and simulating the cell networks. Hence, it has received the greatest attention, not only from the biology community, but also from physics, systems science, etc. A Boolean network is a dynamic logic equation [8].

Roughly speaking, there are two different evolution processes in nature. One is the quantity-based process, which can be described by differential or difference equations. The motions of suns, plants, and satellites, running of machines, cars, etc., are of this type. For this kind of process, people have very good understanding. Another one is logic-based process, which may be described by logic static and/or dynamic equations. Playing games, gambling, control of discrete event systems, etc., are of this type. People are, in general, short of systematic tools to deal with this.

The purpose of this paper is to provide a method to convert logic equations to algebraic equations. Then the methods used for quantity-based process can be used for logic-based process. This paper only considers static equations. We refer to Refs. [9–11] for solving dynamic equations under this framework.

The paper is organized as follows. Section 2 gives a brief survey for semi-tensor product of matrices. The matrix form of logic is explained in Sect. 3. In Sect. 4 the logic equation and its solutions are rigorously defined. Then in Sect. 5 we discuss how to convert a logic equation to a linear system of algebraic equations. In this way, solving a logic equation becomes a trivial task. Section 6 shows how to use this method to solve the logic inference problems. Section 7 is a brief conclusion.

2 Semi-tensor product

This section is a brief introduction to semi-tensor product (STP) of matrices. It plays a fundamental role in the following discussion. We restrict it to the definitions and some related basic properties. In addition, only left semi-tensor product for multiplying dimensional case is involved in the paper. We refer to Refs. [12,13] for right semi-tensor product, arbitrary dimensional case and much more details. Throughout this paper "semi-tensor product" means the left semi-tensor product for multiplying dimensional case.

Definition 1 1) Let $\boldsymbol{X}$ be a row vector of dimension np, and $\boldsymbol{Y}$ be a column vector with dimension p. Then we split $\boldsymbol{X}$ into p equal-size blocks as $\boldsymbol{X}^1, \boldsymbol{X}^2, \ldots, \boldsymbol{X}^p$, which are $1 \times n$ rows. Define the STP, denoted by $\ltimes$, as

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$$\begin{cases} \boldsymbol{X} \ltimes \boldsymbol{Y} = \sum_{i=1}^{p} \boldsymbol{X}^i y_i \in \mathbb{R}^n, \\ \boldsymbol{Y}^{\mathrm{T}} \ltimes \boldsymbol{X}^{\mathrm{T}} = \sum_{i=1}^{p} y_i (\boldsymbol{X}^i)^{\mathrm{T}} \in \mathbb{R}^n. \end{cases} \tag{1}$$

2) Let $\boldsymbol{A} \in \boldsymbol{M}_{m\times n}$ and $\boldsymbol{B} \in \boldsymbol{M}_{p\times q}$. If either n is a factor of p, say $nt=p$ and denote it as $\boldsymbol{A} \prec_t \boldsymbol{B}$, or p is a factor of n, say $n=pt$ and denote it as $\boldsymbol{A} \succ_t \boldsymbol{B}$, then we define the STP of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{C} = \boldsymbol{A} \ltimes \boldsymbol{B}$, in the following: $\boldsymbol{C}$ consists of $m \times q$ blocks as $\boldsymbol{C} = [\boldsymbol{C}^{ij}]$ and each block is

$$\boldsymbol{C}^{ij} = \boldsymbol{A}^i \ltimes \boldsymbol{B}_j, \ \ i = 1,2,...,m, j = 1,2,...,q,$$

where $\boldsymbol{A}^i$ is the ith row of $\boldsymbol{A}$ and $\boldsymbol{B}_j$ is the jth column of $\boldsymbol{B}$.

We use some simple numerical examples to describe it.

Example 1 1) Let

$$\boldsymbol{X} = [1 \quad 2 \quad 3 \quad -1],$$

$$\boldsymbol{Y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then

$$\boldsymbol{X} \ltimes \boldsymbol{Y} = [1 \quad 2]\cdot 1 + [3 \quad -1]\cdot 2 = [7 \quad 0].$$

2) Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Then

$$\boldsymbol{A} \ltimes \boldsymbol{B} = \begin{bmatrix} [1 \quad 2 \quad 1 \quad 1] \ltimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} & [1 \quad 2 \quad 1 \quad 1] \ltimes \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ [2 \quad 3 \quad 1 \quad 2] \ltimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} & [2 \quad 3 \quad 1 \quad 2] \ltimes \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ [3 \quad 2 \quad 1 \quad 0] \ltimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} & [3 \quad 2 \quad 1 \quad 0] \ltimes \begin{bmatrix} -2 \\ -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{bmatrix}.$$

Some fundamental properties of the STP are collected as follows.

Proposition 1 The STP satisfies (as long as the related products are well defined)

1) (Distributive rule)

$$\begin{cases} \boldsymbol{A} \ltimes (\alpha \boldsymbol{B} + \beta \boldsymbol{C}) = \alpha \boldsymbol{A} \ltimes \boldsymbol{B} + \beta \boldsymbol{A} \ltimes \boldsymbol{C}, \\ (\alpha \boldsymbol{B} + \beta \boldsymbol{C}) \ltimes \boldsymbol{A} = \alpha \boldsymbol{B} \ltimes \boldsymbol{A} + \beta \boldsymbol{C} \ltimes \boldsymbol{A}, \end{cases} \alpha,\beta \in \mathbb{R}. \tag{2}$$

2) (Associative rule)

$$\boldsymbol{A} \ltimes (\boldsymbol{B} \ltimes \boldsymbol{C}) = (\boldsymbol{A} \ltimes \boldsymbol{B}) \ltimes \boldsymbol{C}. \tag{3}$$

Proposition 2 Assume $\boldsymbol{A} \succ_k \boldsymbol{B}$. Then

$$\boldsymbol{A} \ltimes \boldsymbol{B} = \boldsymbol{A}(\boldsymbol{B} \otimes \boldsymbol{I}_k). \tag{4}$$

Assume $\boldsymbol{A} \prec_t \boldsymbol{B}$. Then

$$\boldsymbol{A} \ltimes \boldsymbol{B} = (\boldsymbol{A} \otimes \boldsymbol{I}_k)\boldsymbol{B}. \tag{5}$$

Proposition 3 1) Assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are of proper dimensions such that $\boldsymbol{A} \ltimes \boldsymbol{B}$ is defined. Then

$$(\boldsymbol{A} \ltimes \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}} \ltimes \boldsymbol{A}^{\mathrm{T}}. \tag{6}$$

2) In addition, assume that both $\boldsymbol{A}$ and $\boldsymbol{B}$ are invariable. Then

$$(\boldsymbol{A} \ltimes \boldsymbol{B})^{-1} = \boldsymbol{B}^{-1} \ltimes \boldsymbol{A}^{-1}. \tag{7}$$

Proposition 4 Assume that $\boldsymbol{A} \in \boldsymbol{M}_{m\times n}$ is given.

1) Let $\boldsymbol{Z} \in \mathbb{R}^t$ be a row vector. Then

$$\boldsymbol{A} \ltimes \boldsymbol{Z} = \boldsymbol{Z} \ltimes (\boldsymbol{I}_t \otimes \boldsymbol{A}). \tag{8}$$

2) Let $\boldsymbol{Z} \in \mathbb{R}^t$ be a column vector. Then

$$\boldsymbol{Z} \ltimes \boldsymbol{A} = (\boldsymbol{I}_t \otimes \boldsymbol{A}) \ltimes \boldsymbol{Z}. \tag{9}$$

Note that when $\boldsymbol{\xi} \in \mathbb{R}^n$ is a column or a row, $\underbrace{\boldsymbol{\xi} \ltimes \cdots \ltimes \boldsymbol{\xi}}_{k}$ is well defined. We denote it briefly as

$$\boldsymbol{\xi}^k := \underbrace{\boldsymbol{\xi} \ltimes \cdots \ltimes \boldsymbol{\xi}}_{k}.$$

In general, let $\boldsymbol{A} \in \boldsymbol{M}_{m\times n}$ and assume either m is a factor of n or n is a factor of m. Then

$$\boldsymbol{A}^k := \underbrace{\boldsymbol{A} \ltimes \cdots \ltimes \boldsymbol{A}}_{k}$$

is well defined.

Remark 1 Denote by $\boldsymbol{\delta}_m^i$ the ith column of the identity matrix $\boldsymbol{I}_m$ and set $\Delta_m = \{\boldsymbol{\delta}_m^i | i = 1,2,...,m\}$.

A matrix $\boldsymbol{A} \in \boldsymbol{M}_{m\times n}$ is called the logic matrix if $m=2^p$ and $n=2^q$, for some $p,q \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is a set of natural numbers, and the columns of $\boldsymbol{A}$ satisfy

$$\mathrm{Col}(\boldsymbol{A}) \subset \Delta_{2^m}.$$

Denote the set of logic matrices by $\mathcal{L}_{\mathrm{B}}$. Then it is clear that for any $\boldsymbol{A},\boldsymbol{B} \in \mathcal{L}_B$, the semi-tensor product $\boldsymbol{A} \ltimes \boldsymbol{B}$ is always defined. Later on, one will see that in the Boolean network related to matrix forms, the matrices are all in $\mathcal{L}_{\rm B}$. If a matrix $\boldsymbol{A}=[\boldsymbol{\delta}_m^{i_1},\boldsymbol{\delta}_m^{i_2},\ldots,\boldsymbol{\delta}_m^{i_n}]$, then we express it in a condensed format $\boldsymbol{A}=\boldsymbol{\delta}_m[i_1,i_2,\ldots,i_n]$.

Next, we define the swap matrix, which is also called permutation matrix and is defined implicitly in Ref. [14]. Many properties can be found in Ref. [12]. The swap matrix, $\boldsymbol{W}_{[m,n]}$ is an $mn\times mn$ matrix constructed in the following way: label its columns by $[11,12,\ldots,1n,\ldots,m1,m2,\ldots,mn]$ and its rows by $[11,21,\ldots,m1,\ldots,1n,2n,\ldots,mn]$. Then its element in the position $((I,J),(i,j))$ is assigned as

$$w_{(I,J),(i,j)}=\delta_{i,j}^{I,J}=\begin{cases}1, & I=i \text{ and } J=j,\\ 0, & \text{otherwise.}\end{cases} \tag{10}$$

When $m=n$ we briefly denote $\boldsymbol{W}_{[n]}:=\boldsymbol{W}_{[n,n]}$.

Example 2 Let $m=2$ and $n=3$, the swap matrix $\boldsymbol{W}_{[2,3]}$ is constructed as

$$\boldsymbol{W}_{[2,3]}=\begin{array}{c}\begin{array}{cccccc}(11)&(12)&(13)&(21)&(22)&(23)\end{array}\\ \left[\begin{array}{cccccc}1&0&0&0&0&0\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{array}\right]\end{array}\begin{array}{c}\\(11)\\(21)\\(12)\\(22)\\(13)\\(23)\end{array}.$$

Let $\boldsymbol{A}\in\boldsymbol{M}_{m\times n}$, i.e., $\boldsymbol{A}$ is an $m\times n$ matrix. Denote by $\boldsymbol{V}_{\rm r}(\boldsymbol{A})$ the row stacking form of $\boldsymbol{A}$, that is,

$$\boldsymbol{V}_{\rm r}(\boldsymbol{A})=(a_{11},a_{12},\ldots,a_{1n},\ldots,a_{m1},a_{m2},\ldots,a_{mn})^{\rm T},$$

and by $\boldsymbol{V}_{\rm c}(A)$ the column stacking form of $\boldsymbol{A}$, that is,

$$\boldsymbol{V}_{\rm c}(\boldsymbol{A})=(a_{11},a_{21},\ldots,a_{m1},\ldots,a_{1n},a_{2n},\ldots,a_{mn})^{\rm T}.$$

The following "swap" property shows the meaning of the name.

Proposition 5 1) Let $\boldsymbol{X}\in\mathbb{R}^m$ and $\boldsymbol{Y}\in\mathbb{R}^n$ be two columns. Then

$$\begin{cases}\boldsymbol{W}_{[m,n]}\ltimes\boldsymbol{X}\ltimes\boldsymbol{Y}=\boldsymbol{Y}\ltimes\boldsymbol{X},\\ \boldsymbol{W}_{[n,m]}\ltimes\boldsymbol{Y}\ltimes\boldsymbol{X}=\boldsymbol{X}\ltimes\boldsymbol{Y}.\end{cases} \tag{11}$$

2) Let $\boldsymbol{A}\in\boldsymbol{M}_{m\times n}$. Then

$$\begin{cases}\boldsymbol{W}_{[m,n]}\boldsymbol{V}_{\rm r}(\boldsymbol{A})=\boldsymbol{V}_{\rm c}(\boldsymbol{A}),\\ \boldsymbol{W}_{[n,m]}\boldsymbol{V}_{\rm c}(\boldsymbol{A})=\boldsymbol{V}_{\rm r}(\boldsymbol{A}).\end{cases} \tag{12}$$

3) Let $\boldsymbol{X}_i\in\mathbb{R}^{n_i}$, $i=1,2,\ldots,m$. Then

$$\begin{aligned}&(\boldsymbol{I}_{n_1+\cdots+n_{k-1}}\otimes\boldsymbol{W}_{[n_k,n_{k+1}]}\otimes\boldsymbol{I}_{n_{k+2}+\cdots+n_m})\boldsymbol{X}_1\ltimes\cdots\ltimes\boldsymbol{X}_k\\ &\quad\ltimes\boldsymbol{X}_{k+1}\ltimes\cdots\ltimes\boldsymbol{X}_m=\boldsymbol{X}_1\ltimes\cdots\ltimes\boldsymbol{X}_{k+1}\ltimes\boldsymbol{X}_k\ltimes\cdots\ltimes\boldsymbol{X}_m.\end{aligned} \tag{13}$$

Proposition 6 The swap matrix is an orthogonal matrix as

$$\boldsymbol{W}_{[m,n]}^{\rm T}=\boldsymbol{W}_{[m,n]}^{-1}=\boldsymbol{W}_{[n,m]}. \tag{14}$$

Proposition 7

$$\boldsymbol{W}_{[pq,r]}=(\boldsymbol{W}_{[p,r]}\otimes\boldsymbol{I}_q)(\boldsymbol{I}_p\otimes\boldsymbol{W}_{[q,r]}). \tag{15}$$

Taking transposition on both sides of Eq. (15) yields

$$\boldsymbol{W}_{[r,pq]}=(\boldsymbol{I}_p\otimes\boldsymbol{W}_{[r,q]})(\boldsymbol{W}_{[r,p]}\otimes\boldsymbol{I}_q). \tag{16}$$

The swap matrix can also be constructed in the following way.

Proposition 8

$$W_{[m,n]}=(\boldsymbol{\delta}_n^1\ltimes\boldsymbol{\delta}_m^1\ \cdots\ \boldsymbol{\delta}_n^n\ltimes\boldsymbol{\delta}_m^1\ \cdots\ \boldsymbol{\delta}_n^1\ltimes\boldsymbol{\delta}_m^m\ \cdots\ \boldsymbol{\delta}_n^n\ltimes\boldsymbol{\delta}_m^m). \tag{17}$$

In Ref. [14], Eq. (17) is used as the definition.

Remark 2 It is obvious that if $\boldsymbol{A}\in\boldsymbol{M}_{m\times s}$ and $\boldsymbol{B}\in\boldsymbol{M}_{s\times n}$, i.e., the conventional matrix product $\boldsymbol{AB}$ exists, then

$$\boldsymbol{AB}=\boldsymbol{A}\ltimes\boldsymbol{B}.$$

Hence, the semi-tensor product is a generalization of conventional matrix product. Based on this, the notation "$\ltimes$" can be omitted. In the following, all the matrix products are assumed to be semi-tensor product and the notation "$\ltimes$" is always omitted. As the conventional matrix product exists, the product turns to be conventional matrix product automatically.

3 Matrix expression of logic

In this section, we consider the matrix expression of logic. Under matrix expression, a logical variable is expressed as a vector and an r-ary logical operator is expressed by a 2×2^r matrix, called the structure matrix of the operator. Then the logical action of the operator over r logical variables becomes a matrix product of the structure matrix with r vectors. We refer to Refs. [13,15,16] for details.

First, we give some necessary notations and concerning results for logic.

Definition 2 1) A logical domain, denoted by D, is defined as

$$D=\{T=1,F=0\}. \tag{18}$$

2) An n-ary logical operator is a function $t:D^n\rightarrow D$. n is called the arity of t, denoted by $ar(t)=n$ [17].

To use matrix expression we identify each element in D with a vector as $T\sim\begin{bmatrix}1\\0\end{bmatrix}$ and $F\sim\begin{bmatrix}0\\1\end{bmatrix}$, and denote

$$D_{\rm v}=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

Using this vector expression, we can define the structure matrix of a logical operator.

Definition 3 A 2×2^s matrix $\boldsymbol{M}_\sigma$ is called the structure matrix of an s-ary logical operator σ, if

$$\sigma(\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_s) = \boldsymbol{M}_\sigma \boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_s, \tag{19}$$

where $\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_s \in D_{\rm v}$.

If such a matrix exists, it uniquely determines the logic operator. To show the existence of such a matrix for each logical operator, we need some preparations. Define a matrix, called the power-reducing matrix, as

$$\boldsymbol{M}_{\rm r} = \boldsymbol{\delta}_4[1 \quad 4]. \tag{20}$$

Its name is from the following property.

Lemma 1 Let $\boldsymbol{P} \in D_{\rm v}$. Then we have

$$\boldsymbol{P}^2 = \boldsymbol{M}_{\rm r}\boldsymbol{P}. \tag{21}$$

In a logical expression a logical variable is constant if its value is assigned in advance, it is called an argument if its value is variable. Using this concept and above lemma, we can easily prove the following.

Theorem 1 Any logical function $L(\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_s)$ with logical arguments $\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_s \in D_{\rm v}$ can be expressed in a canonical form as

$$L(\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_s) = \boldsymbol{M}_L \boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_s, \tag{22}$$

where $\boldsymbol{M}_L$ is a 2×2^s matrix, called the structure matrix of L.

Next, we give some examples to illustrate structure matrix.

Example 3 1) Consider a fundamental unary operator: Negation, $\neg\boldsymbol{P}$, and four fundamental binary operators [18]: Disjunction, $\boldsymbol{P}\vee\boldsymbol{Q}$; Conjunction, $\boldsymbol{P}\wedge\boldsymbol{Q}$; Implication, $\boldsymbol{P}\rightarrow\boldsymbol{Q}$; Equivalence, $\boldsymbol{P}\leftrightarrow\boldsymbol{Q}$. Their structure matrices are as follows:

$$\begin{aligned}
\boldsymbol{M}_\neg &:= \boldsymbol{M}_{\rm n} = \boldsymbol{\delta}_2[2 \quad 1],\\
\boldsymbol{M}_\vee &:= \boldsymbol{M}_{\rm d} = \boldsymbol{\delta}_2[1 \quad 1 \quad 1 \quad 2],\\
\boldsymbol{M}_\wedge &:= \boldsymbol{M}_{\rm c} = \boldsymbol{\delta}_2[1 \quad 2 \quad 2 \quad 2],\\
\boldsymbol{M}_\rightarrow &:= \boldsymbol{M}_{\rm i} = \boldsymbol{\delta}_2[1 \quad 2 \quad 1 \quad 1],\\
\boldsymbol{M}_\leftrightarrow &:= \boldsymbol{M}_{\rm e} = \boldsymbol{\delta}_2[1 \quad 2 \quad 2 \quad 1].
\end{aligned} \tag{23}$$

2) Assume

$$L(\boldsymbol{P},\boldsymbol{Q}) = (\boldsymbol{P}\rightarrow\boldsymbol{Q})\vee(\neg\boldsymbol{P}).$$

Using vector form of logic variables, Proposition 4 and the order-reducing matrix, we have

$$\begin{aligned}
L(\boldsymbol{P},\boldsymbol{Q}) &= \boldsymbol{M}_{\rm d}(\boldsymbol{M}_{\rm i}\boldsymbol{P}\boldsymbol{Q})(\boldsymbol{M}_{\rm n}\boldsymbol{P})\\
&= \boldsymbol{M}_{\rm d}\boldsymbol{M}_{\rm i}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\rm n})\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}\\
&= \boldsymbol{M}_{\rm d}\boldsymbol{M}_{\rm i}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\rm n})\boldsymbol{P}\boldsymbol{W}_{[2]}\boldsymbol{P}\boldsymbol{Q}\\
&= \boldsymbol{M}_{\rm d}\boldsymbol{M}_{\rm i}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\rm n})(\boldsymbol{I}_2\otimes\boldsymbol{M}_{[2]})\boldsymbol{P}^2\boldsymbol{Q}.
\end{aligned}$$

We conclude that

$$\boldsymbol{M}_L = \boldsymbol{M}_{\rm d}\boldsymbol{M}_{\rm i}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\rm n})(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2]})\boldsymbol{M}_{\rm r} = \boldsymbol{\delta}_2[1 \quad 2 \quad 1 \quad 1].$$

Remark 3 1) In fact, there are 2^{2^r} r-ary logical operators.

2) For any binary logical operator σ, we have

$$\boldsymbol{P}\sigma\boldsymbol{Q} = \boldsymbol{M}_\sigma\boldsymbol{P}\boldsymbol{Q}, \quad \boldsymbol{P},\boldsymbol{Q} \in D_{\rm v}.$$

3) In the study of Boolean Networks mod 2 algebra is used. It is easy to figure out that the mod 2 product is exactly the conjunction. That is,

$$\boldsymbol{P}\times\boldsymbol{Q}\,(\text{mod } 2) = \boldsymbol{P}\wedge\boldsymbol{Q}, \quad \boldsymbol{P},\boldsymbol{Q}\in D. \tag{24}$$

The mod 2 addition can be expressed as

$$\boldsymbol{P}+\boldsymbol{Q}\,(\text{mod } 2) = \neg(\boldsymbol{P}\leftrightarrow\boldsymbol{Q}), \quad \boldsymbol{P},\boldsymbol{Q}\in D. \tag{25}$$

So its structure matrix is

$$\boldsymbol{M}_+ = \boldsymbol{M}_{\rm n}\boldsymbol{M}_{\rm e} = \boldsymbol{\delta}_2[2 \quad 1 \quad 1 \quad 2]. \tag{26}$$

In logic it is called the Exclusive OR [18].

In the following we use D and $D_{\rm v}$ alternatively for logical variables $\boldsymbol{P}$, $\boldsymbol{Q}$, etc., without explanation. From the contents it is easy to figure out which form is used then.

4 Solutions to logic equations

A logic variable p is called a logic argument or logic unknown if it can take a value $p\in D=\{T,F\}$ to suit a certain logic requirement. A logic constant c is an invariant value $c\in D$.

Definition 4 A standard logic equation is expressed as

$$\begin{cases}
f_1(p_1,p_2,\ldots,p_n) = c_1,\\
f_2(p_1,p_2,\ldots,p_n) = c_2,\\
\qquad\cdots\\
f_m(p_1,p_2,\ldots,p_n) = c_m,
\end{cases} \tag{27}$$

where f_i, $i=1,2,\ldots,m$, are logic functions; p_i, $i=1,2,\ldots,n$, are logic arguments (unknowns); c_i, $i=1,2,\ldots,m$, are logic constants. A set of logic constants d_i, $i=1,2,\ldots,n$, which make

$$p_i = d_i, \quad i=1,2,\ldots,n, \tag{28}$$

satisfy Eq. (27), is said to be a solution to logic Eq. (27).

Example 4 Consider the following system:

$$\begin{cases}
p\wedge q = c_1,\\
q\vee r = c_2,\\
r\leftrightarrow(\neg p) = c_3.
\end{cases} \tag{29}$$

1) Assume the logic constants are $c_1=1$, $c_2=1$, $c_3=1$. A straightforward verification shows that

$$\begin{cases} p = 1, \\ q = 1, \\ r = 0 \end{cases}$$

is the only solution.

2) Assume the logic constants are $c_1 = 1$, $c_2 = 0$, $c_3 = 1$. Then one can check that there is no solution.

3) Assume the logic constants are $c_1 = 0$, $c_2 = 1$, $c_3 = 0$. Then there are two solutions

$$\begin{cases} p_1 = 1, \\ q_1 = 0, \\ r_1 = 1, \end{cases}$$

and

$$\begin{cases} p_1 = 0, \\ q_1 = 1, \\ r_1 = 0. \end{cases}$$

Example 4 is heuristic. It shows that the solutions of logic equations are quite different from the ones of linear algebraic equations where the type of equations depends only on the coefficients of the systems.

5 Convert to algebraic equation

This section considers how to solve logic Eq. (27). The basic idea is: first, convert the logic Eq. (27) into a linear algebraic equation, then solve the algebraic equation. To do this, we need some preparations.

Lemma 2 Let p_i, $i = 1,2,\ldots,n$, be logic variables. We define

$$\boldsymbol{x} = \ltimes_{i=1}^{n} p_i.$$

Then p_i, $i = 1,2,\ldots,n$, are uniquely determined by $\boldsymbol{x}$.

Proof We prove this by giving a formula to calculate p_i. First of all, since $p_i \in \Delta_2$, it follows that $\boldsymbol{x} \in \Delta_{2^n}$. Now we can assume $\boldsymbol{x} = \boldsymbol{\delta}_{2^n}^{i}$. Split $\boldsymbol{x}$ into two equal segments as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix},$$

where either $\boldsymbol{x}_1 \in \Delta_{2^{n-1}}$ and $\boldsymbol{x}_2 = \boldsymbol{0}_{2^{n-1}}$ or $\boldsymbol{x}_1 = \boldsymbol{0}_{2^{n-1}}$ and $\boldsymbol{x}_2 \in \Delta_{2^{n-1}}$. According to the definition of semi-tensor product, if $\boldsymbol{x}_2 = \boldsymbol{0}_{2^{n-1}}$ then $p_1 = 1$, and if $\boldsymbol{x}_1 = \boldsymbol{0}_{2^{n-1}}$ then $p_1 = 0$. Then we can split non-zero segment, say $\boldsymbol{x}_1 \neq \boldsymbol{0}_{2^{n-1}}$, into two equal parts as

$$\boldsymbol{x}_1 = \begin{bmatrix} \boldsymbol{x}_{11} \\ \boldsymbol{x}_{12} \end{bmatrix},$$

and do the same judgment for p_2, and so on. The result follows.

Based on the argument in the proof of the last lemma, we give an algorithm as follows:

Algorithm 1 Let $\ltimes_{k=1}^{n} p_k = \boldsymbol{\delta}_{2^n}^{i}$, where $p_k \in \Delta_2$ are in vector form. Then

1) $\{p_k\}$ can be calculated from i inductively by the following method:

Set

$$q_0 : 2^n - i.$$

Calculate p_k and q_k, $k = 1,2,\ldots,n$, (in scalar form) recursively by

$$\begin{cases} p_k = \left[\dfrac{q_{k-1}}{2^{n-k}}\right], \\ \\ q_k = q_{k-1} - p_k 2^{n-k}, \end{cases} \quad k = 1,2,\ldots,n, \tag{30}$$

where in the first equation $[a]$ is the largest integer less than or equal to a.

2) i can be calculated from $\{p_k\}$ (in scalar form) by

$$i = \sum_{k=1}^{n} (1 - p_k) 2^{n-k} + 1. \tag{31}$$

We give an example to show this.

Example 5 Assume $\boldsymbol{x} = p_1 p_2 p_3 p_4 p_5$ and the value of $\boldsymbol{x}$ is known as $\boldsymbol{x} = \boldsymbol{\delta}_{32}^{7}$. Then we try to get the values of p_i, $i = 1,2,\ldots,5$. Using Algorithm 1, we have

$$q_0 = 2^5 - 7 = 32 - 7 = 25.$$

It follows that

$$\begin{aligned} &p_1 = [q_0/16] = 1, && q_1 = q_0 - p_1 \times 16 = 9, \\ &p_2 = [q_1/8] = 1, && q_2 = q_1 - p_2 \times 8 = 1, \\ &p_3 = [q_2/4] = 0, && q_3 = q_2 - p_3 \times 4 = 1, \\ &p_4 = [q_3/2] = 0, && q_4 = q_3 - p_4 \times 2 = 1, \\ &p_5 = [q_4/1] = 1. && \end{aligned}$$

We conclude that $p_1 = 1 \sim \boldsymbol{\delta}_2^1$, $p_2 = 1 \sim \boldsymbol{\delta}_2^1$, $p_3 = 0 \sim \boldsymbol{\delta}_2^2$, $p_4 = 0 \sim \boldsymbol{\delta}_2^2$, and $p_5 = 1 \sim \boldsymbol{\delta}_2^1$.

Next, we construct a matrix, which may be called the group power reducing matrix, as follows. For $k \geqslant 1$, define

$$\boldsymbol{\Phi}_k = \prod_{i=1}^{k} \boldsymbol{I}_{2^{i-1}} \otimes \left[(\boldsymbol{I}_2 \otimes \boldsymbol{W}_{[2,2^{k-i}]}) \boldsymbol{M}_{\mathrm{r}} \right]. \tag{32}$$

Then we have

Lemma 3 Assume $z_k = p_1 p_2 \cdots p_k$, where $p_i \in \Delta_2$, $i = 1,2,\ldots,k$, then

$$z_k^2 = \boldsymbol{\Phi}_k z_k. \tag{33}$$

Proof It is proved by mathematical induction. When $k = 1$, using Lemma 1

$$z_1^2 = p_1^2 = \boldsymbol{M}_{\mathrm{r}} p_1.$$

In the above formula,

$$\boldsymbol{\Phi}_1 = (\boldsymbol{I}_2 \otimes \boldsymbol{W}_{[2,1]})\boldsymbol{M}_{\mathrm{r}}.$$

Note that $\boldsymbol{W}_{[2,1]}=\boldsymbol{I}_2$, it follows that $\boldsymbol{\Phi}_1 = \boldsymbol{M}_{\mathrm{r}}$. Hence, Eq. (33) is true for $k=1$. Assume Eq. (33) is true for $k=s$, then for $k=s+1$ we have

$$\begin{aligned}\boldsymbol{P}_{s+1}^2 &= \boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+1}\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+1}\\ &= \boldsymbol{A}_1\boldsymbol{W}_{[2,2^k]}\boldsymbol{A}_1(\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+1})^2\\ &= \left(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,2^k]}\right)\boldsymbol{A}_1^2(\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+1})^2\\ &= \left[\left(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,2^k]}\right)\boldsymbol{M}_{\mathrm{r}}\right]\boldsymbol{A}_1(\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+1})^2.\end{aligned}$$

Using induction assumption to the last fact of the above expression, we have

$$\begin{aligned}z_{s+1}^2 &= \left(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,2^s]}\right)\boldsymbol{M}_{\mathrm{r}}p_1\left(\prod_{i=1}^{s}\boldsymbol{I}_{2^{i-1}}\otimes\left[\left(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,2^{s-i}]}\right)\boldsymbol{M}_{\mathrm{r}}\right]\right)\\ &\quad\ltimes p_2p_3\cdots p_{k+1}\\ &= \left[\left(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,2^s]}\right)\boldsymbol{M}_{\mathrm{r}}\right]\left(\prod_{i=1}^{s}\boldsymbol{I}_{2^i}\otimes\left[\left(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,2^{s-i}]}\right)\boldsymbol{M}_{\mathrm{r}}\right]\right)\\ &\quad\ltimes p_1p_2\cdots p_{k+1}.\end{aligned}$$

The conclusion follows.

Before presenting another lemma we have to introduce another concept, which is a dummy operator, σ_{d}, by dummy operator

$$\sigma_{\mathrm{d}}(p,q) = q, \quad \forall p,q \in D. \tag{34}$$

It is easy to figure out that the structure matrix of the dummy operator is

$$\boldsymbol{E}_{\mathrm{d}} := \boldsymbol{\delta}_2[1 \quad 2 \quad 1 \quad 2]. \tag{35}$$

It follows from definition that for any two logic variables $\boldsymbol{X}$ and $\boldsymbol{Y}$,

$$\boldsymbol{E}_{\mathrm{d}}\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y} \ \text{ or } \ \boldsymbol{E}_{\mathrm{d}}\boldsymbol{W}_{[2]}\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{X}. \tag{36}$$

Lemma 4 Denote $\boldsymbol{x} = \ltimes_{i=1}^n p_i$. Using vector form, each logic equation

$$f_i(p_1,p_2,\ldots,p_n) = c_i, \quad i = 1,2,\ldots,m,$$

in Eq. (27) can be expressed as

$$\boldsymbol{M}_i\boldsymbol{x} = c_i, \quad i = 1,2,\ldots,m, \tag{37}$$

where $\boldsymbol{M}_i \in \boldsymbol{M}_{2^m\times 2^n}$ are Boolean matrixes.

Proof Assume f_i is a logic equation on $p_1,p_2,\ldots,p_n$. Let $\boldsymbol{M}_i$ be the structure matrix of f_i. Then we have Eq. (37) immediately. Assume some $p_{j'}$s do not appear into f_i. Using the dummy operator technique, we can still get Eq. (37) by adding dummy variables.

Now we are ready to present the main result, which converts logic Eq. (27) into an algebraic equation.

Theorem 2 Let $\boldsymbol{x} = \ltimes_{i=1}^n p_i$, $\boldsymbol{b} = \ltimes_{i=1}^m c_i$. Then the logic Eq. (27) can be converted into a linear algebraic equation as

$$\boldsymbol{L}\boldsymbol{x} = \boldsymbol{b}, \tag{38}$$

where $\boldsymbol{M}_i$ are defined in Eq. (37) and

$$\boldsymbol{L} = \boldsymbol{M}_1 \ltimes_{j=2}^n[(\boldsymbol{I}_2\otimes\boldsymbol{M}_j)\boldsymbol{\Phi}_n]. \tag{39}$$

Proof Note that from Lemma 3 we have

$$\boldsymbol{x}(t)^2 = \boldsymbol{\Phi}_n\boldsymbol{x}(t).$$

Multiplying Eq. (37) together yields

$$\begin{aligned}\boldsymbol{b} &= \boldsymbol{M}_1\boldsymbol{x}(t)\boldsymbol{M}_2\boldsymbol{x}(t)\cdots\boldsymbol{M}_n\boldsymbol{x}(t)\\ &= \boldsymbol{M}_1(\boldsymbol{I}_2\otimes\boldsymbol{M}_2)\boldsymbol{x}(t)^2\boldsymbol{M}_3\boldsymbol{x}(t)\cdots\boldsymbol{M}_n\boldsymbol{x}(t)\\ &= \boldsymbol{M}_1(\boldsymbol{I}_2\otimes\boldsymbol{M}_2)\boldsymbol{\Phi}_n\boldsymbol{x}(t)\boldsymbol{M}_3\boldsymbol{x}(t)\cdots\boldsymbol{M}_n\boldsymbol{x}(t)\\ &= \cdots\\ &= \boldsymbol{M}_1(\boldsymbol{I}_2\otimes\boldsymbol{M}_2)\boldsymbol{\Phi}_n(\boldsymbol{I}_2\otimes\boldsymbol{M}_3)\boldsymbol{\Phi}_n\cdots(\boldsymbol{I}_2\otimes\boldsymbol{M}_n)\boldsymbol{\Phi}_n\boldsymbol{x}(t).\end{aligned}$$

Eq. (39) follows.

Remark 4 1) For a particular logic equation to get its algebraic form, we need not use Eq. (39). In most cases, $\boldsymbol{L}$ can be obtained by a direct computation.

2) Using Lemma 2, as long as $\boldsymbol{x}$ is solved from algebraic Eq. (38), the logic unknown p_i, $i=1,2,\ldots,n$, are easily solvable.

3) In Eq. (38), the coefficient matrix $\boldsymbol{L} \in \boldsymbol{M}_{2^m\times 2^n}$ is a logic matrix, and the constant vector $\boldsymbol{b} \in \Delta_{2^m}$.

Since $\boldsymbol{L} \in \mathcal{L}_B$ and $\boldsymbol{b} \in \Delta_{2^m}$, it is clear that algebraic Eq. (38) has solution $\boldsymbol{x} \in \Delta_{2^n}$, if and only if $\boldsymbol{b} \in \mathrm{Col}(\boldsymbol{L})$.

Express $\boldsymbol{L}$ in condensed form as $\boldsymbol{L} = \boldsymbol{\delta}_{2^m}[i_1,i_2,\ldots,i_{2^n}]$, we define a set

$$\Lambda = \{\lambda \mid \boldsymbol{\delta}_{2^m}^{i_\lambda} = \boldsymbol{b},\ 1 \leqslant \lambda \leqslant 2^n\}.$$

Then the following result is obvious:

Theorem 3 Using the above notation, the solution of Eq. (38) is

$$\boldsymbol{x} = \boldsymbol{\delta}_{2^n}^{\lambda}, \quad \lambda \in \Lambda. \tag{40}$$

As an application, we re-consider Example 4.

Example 6 Consider the system (29) again. We have its algebraic form as

$$\begin{cases}\boldsymbol{M}_{\mathrm{c}}pq = c_1,\\ \boldsymbol{M}_{\mathrm{d}}qr = c_2,\\ \boldsymbol{M}_{\mathrm{e}}r(\boldsymbol{M}_{\mathrm{n}}p) = c_3.\end{cases} \tag{41}$$

Multiplying three equations together yields

$$\boldsymbol{M}_{\mathrm{c}}pq\boldsymbol{M}_{\mathrm{d}}qr\boldsymbol{M}_{\mathrm{e}}r\boldsymbol{M}_{\mathrm{n}}p = c_1c_2c_3 := \boldsymbol{b}. \tag{42}$$

Next, we simplify the left hand side of Eq. (42):

$$\begin{aligned}
&\boldsymbol{M}_{\mathrm{c}}pq\boldsymbol{M}_{\mathrm{d}}qr\boldsymbol{M}_{\mathrm{e}}r\boldsymbol{M}_{\mathrm{n}}p \\
&\quad = \boldsymbol{M}_{\mathrm{c}}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{d}})pq^2r\boldsymbol{M}_{\mathrm{e}}r\boldsymbol{M}_{\mathrm{n}}p \\
&\quad = \boldsymbol{M}_{\mathrm{c}}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{d}})(\boldsymbol{I}_{16}\otimes\boldsymbol{M}_{\mathrm{e}})pq^2r^2\boldsymbol{M}_{\mathrm{n}}p \\
&\quad = \boldsymbol{M}_{\mathrm{c}}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{d}})(\boldsymbol{I}_{16}\otimes\boldsymbol{M}_{\mathrm{e}})(\boldsymbol{I}_{32}\otimes\boldsymbol{M}_{\mathrm{n}})pq^2r^2p \\
&\quad = \boldsymbol{M}_{\mathrm{c}}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{d}})(\boldsymbol{I}_{16}\otimes\boldsymbol{M}_{\mathrm{e}})(\boldsymbol{I}_{32}\otimes\boldsymbol{M}_{\mathrm{n}})p\boldsymbol{W}_{[2,16]}pq^2r^2 \\
&\quad = \boldsymbol{M}_{\mathrm{c}}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{d}})(\boldsymbol{I}_{16}\otimes\boldsymbol{M}_{\mathrm{e}})(\boldsymbol{I}_{32}\otimes\boldsymbol{M}_{\mathrm{n}})(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,16]}) \\
&\qquad \ltimes p^2q^2r^2 \\
&\quad = \boldsymbol{M}_{\mathrm{c}}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{d}})(\boldsymbol{I}_{16}\otimes\boldsymbol{M}_{\mathrm{e}})(\boldsymbol{I}_{32}\otimes\boldsymbol{M}_{\mathrm{n}})(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,16]}) \\
&\qquad \ltimes \boldsymbol{M}_{\mathrm{r}}p\boldsymbol{M}_{\mathrm{r}}q\boldsymbol{M}_{\mathrm{r}}r \\
&\quad = \boldsymbol{M}_{\mathrm{c}}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{d}})(\boldsymbol{I}_{16}\otimes\boldsymbol{M}_{\mathrm{e}})(\boldsymbol{I}_{32}\otimes\boldsymbol{M}_{\mathrm{n}})(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,16]}) \\
&\qquad \ltimes \boldsymbol{M}_{\mathrm{r}}(\boldsymbol{I}_2\otimes\boldsymbol{M}_{\mathrm{r}})(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{r}})pqr \\
&\quad := \boldsymbol{L}\boldsymbol{x}.
\end{aligned}$$

It is easy to calculate that

$$\begin{aligned}
\boldsymbol{L} &= \boldsymbol{M}_{\mathrm{c}}(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{d}})(\boldsymbol{I}_{16}\otimes\boldsymbol{M}_{\mathrm{e}})(\boldsymbol{I}_{32}\otimes\boldsymbol{M}_{\mathrm{n}}) \\
&\quad \ltimes (\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,16]})\boldsymbol{M}_{\mathrm{r}}(\boldsymbol{I}_2\otimes\boldsymbol{M}_{\mathrm{r}})(\boldsymbol{I}_4\otimes\boldsymbol{M}_{\mathrm{r}}) \\
&= \boldsymbol{\delta}_8[2\quad 1\quad 6\quad 7\quad 5\quad 6\quad 5\quad 8].
\end{aligned}$$

Now let $\boldsymbol{b} = \boldsymbol{\delta}_8^1$. Then $\Lambda = \{2\}$, that is, the second column of $\boldsymbol{L}$ equals $\boldsymbol{b}$. The solution is $\boldsymbol{x} = \boldsymbol{\delta}_8^2$. Back to Boolean form we have:

$$\boldsymbol{b} = \boldsymbol{\delta}_8^1, \text{ iff, } c_1 = 1,\ c_2 = 1,\ c_3 = 1;$$

$$\boldsymbol{x} = \boldsymbol{\delta}_8^2, \text{ iff, } p_1 = 1,\ p_2 = 1,\ p_3 = 0.$$

We list all possible constants and their corresponding solutions in Table 1.

Table 1 Solutions of Eq. (29)

$\boldsymbol{b}$	(c_1,c_2,c_3)	Λ	$\boldsymbol{x}$	(p_1,p_2,p_3)
$\boldsymbol{\delta}_8^1$	(1,1,1)	{2}	$\boldsymbol{\delta}_8^2$	(1,1,0)
$\boldsymbol{\delta}_8^2$	(1,1,0)	{1}	$\boldsymbol{\delta}_8^1$	(1,1,1)
$\boldsymbol{\delta}_8^3$	(1,0,1)	$\boldsymbol{\Phi}$	—	—
$\boldsymbol{\delta}_8^4$	(1,0,0)	$\boldsymbol{\Phi}$	—	—
$\boldsymbol{\delta}_8^5$	(0,1,1)	{5,7}	$\boldsymbol{\delta}_8^5$, $\boldsymbol{\delta}_8^7$	(0,1,0), (0,0,1)
$\boldsymbol{\delta}_8^6$	(0,1,0)	{3,6}	$\boldsymbol{\delta}_8^3$, $\boldsymbol{\delta}_8^6$	(1,0,1), (0,1,0)
$\boldsymbol{\delta}_8^7$	(0,0,1)	{4}	$\boldsymbol{\delta}_8^4$	(1,0,0)
$\boldsymbol{\delta}_8^8$	(0,0,0)	{8}	$\boldsymbol{\delta}_8^8$	(0,0,0)

When the number of unknowns is not very small, calculating the coefficient matrix by paper and hand will be very difficult. A simple routine can do this easily. We give another example.

Example 7 Consider the following logic system:

$$\begin{cases}
p_1 \wedge p_2 = c_1, \\
p_2 \vee (p_3 \leftrightarrow p_2) = c_2, \\
p_5 \rightarrow (p_4 \vee p_3) = c_3, \\
\neg p_3 = c_4, \\
p_4 \vee (p_5 \wedge p_2) = c_5, \\
(p_6 \vee p_2) \wedge p_6) = c_6, \\
(\neg p_{10}) \rightarrow p_7 = c_7, \\
p_5 \wedge p_6 \wedge p_7 = c_8, \\
(p_6 \vee p_9) \leftrightarrow p_3 = c_9.
\end{cases} \tag{43}$$

Its algebraic form is

$$\begin{cases}
\boldsymbol{M}_{\mathrm{c}}p_1p_2 = c_1, \\
\boldsymbol{M}_{\mathrm{d}}p_2\boldsymbol{M}_{\mathrm{e}}p_3p_2 = c_2, \\
\boldsymbol{M}_{\mathrm{i}}p_5\boldsymbol{M}_{\mathrm{d}}p_4p_3 = c_3, \\
\boldsymbol{M}_{\mathrm{n}}p_3 = c_4, \\
\boldsymbol{M}_{\mathrm{d}}p_4\boldsymbol{M}_{\mathrm{c}}p_5p_2 = c_5, \\
\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{d}}p_6p_2p_6 = c_6, \\
\boldsymbol{M}_{\mathrm{i}}\boldsymbol{M}_{\mathrm{n}}p_{10}p_7 = c_7, \\
\boldsymbol{M}_{\mathrm{c}}^2p_5p_6p_7 = c_8, \\
\boldsymbol{M}_{\mathrm{e}}\boldsymbol{M}_{\mathrm{d}}p_6p_9p_3 = c_9.
\end{cases} \tag{44}$$

Of course, we can convert Eq. (44) into an algebraic equation as

$$\boldsymbol{L}\ltimes_{i=1}^9 p_i = \ltimes_{i=1}^9 c_i,$$

or $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{b}$.

Then $\boldsymbol{L} \in \boldsymbol{M}_{512\times 1024}$. To save space, the first and last few columns (in condensed form) are given as follows:

$$\begin{aligned}
\boldsymbol{\delta}_{2^9}[&33\ 33\ 33\ 33\ 35\ 39\ 35\ 39\ 43\ 43\ 44\ 44\ 43\ 47\ 44\ 48 \\
&35\ 35\ 35\ 35\ 35\ 39\ 35\ 39\ 43\ 43\ 44\ 44\ 43\ 47\ 44\ 48 \\
&33\ 33\ 33\ 33\ 35\ 39\ 35\ 39\ 43\ 43\ 44\ 44\ 43\ 47\ 44\ 48 \\
&51\ 51\ 51\ 51\ 51\ 55\ 51\ 55\ 59\ 59\ 60\ 60\ 59\ 63\ 60\ 64 \\
&2\ 2\ 2\ 2\ 4\ 8\ 4\ 8\ 12\ 12\ 11\ 11\ 12\ 16\ 11\ 15 \\
&\cdots \\
&268\ 272\ 267\ 271\ 260\ 260\ 260\ 260\ 260\ 264\ 260\ 264\ 268 \\
&268\ 267\ 267\ 268\ 267\ 267\ 271\ 338\ 338\ 338\ 338\ 340\ 344 \\
&340\ 344\ 348\ 348\ 347\ 347\ 348\ 352\ 347\ 351\ 276\ 276\ 276 \\
&276\ 276\ 280\ 276\ 284\ 284\ 284\ 283\ 283\ 284\ 288\ 283\ 287].
\end{aligned}$$

Next, give a special set of logic constants, we solve the logic equation. Assume $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $c_4 = 0$, $c_5 = 1$, $c_6 = 1$, $c_7 = 1$, $c_8 = 0$, $c_9 = 1$. Then

$$\boldsymbol{b} = \ltimes_{i=1}^9 c_i = \boldsymbol{\delta}_{2^9}^{35}.$$

Next, we can find the set of Λ, such that the columns of $\boldsymbol{L}$, $\boldsymbol{L}_\lambda = \boldsymbol{b}$, $\lambda \in \Lambda$. It is easy to calculate by computer that

$$\Lambda = \{5,7,17,18,19,20,21,23,37,39\}.$$

According to Theorem 3, there are 10 corresponding solutions, which can be easily calculated as

1) $\boldsymbol{x}_1 = \boldsymbol{\delta}_{2^9}^5$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 1, p_5 = 1, p_6 = 1, \\ p_7 = 0, p_8 = 1, p_9 = 1. \end{cases}$$

2) $\boldsymbol{x}_2 = \boldsymbol{\delta}_{2^9}^7$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 1, p_5 = 1, p_6 = 1, \\ p_7 = 0, p_8 = 0, p_9 = 1. \end{cases}$$

3) $\boldsymbol{x}_3 = \boldsymbol{\delta}_{2^9}^{17}$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 1, p_5 = 0, p_6 = 1, \\ p_7 = 1, p_8 = 1, p_9 = 1. \end{cases}$$

4) $\boldsymbol{x}_4 = \boldsymbol{\delta}_{2^9}^{18}$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 1, p_5 = 0, p_6 = 1, \\ p_7 = 1, p_8 = 1, p_9 = 0. \end{cases}$$

5) $\boldsymbol{x}_5 = \boldsymbol{\delta}_{2^9}^{19}$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 1, p_5 = 0, p_6 = 1, \\ p_7 = 1, p_8 = 0, p_9 = 1. \end{cases}$$

6) $\boldsymbol{x}_6 = \boldsymbol{\delta}_{2^9}^{20}$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 1, p_5 = 0, p_6 = 1, \\ p_7 = 1, p_8 = 0, p_9 = 0. \end{cases}$$

7) $\boldsymbol{x}_7 = \boldsymbol{\delta}_{2^9}^{21}$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 1, p_5 = 0, p_6 = 1, \\ p_7 = 0, p_8 = 1, p_9 = 1. \end{cases}$$

8) $\boldsymbol{x}_8 = \boldsymbol{\delta}_{2^9}^{23}$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 1, p_5 = 0, p_6 = 1, \\ p_7 = 0, p_8 = 0, p_9 = 1. \end{cases}$$

9) $\boldsymbol{x}_9 = \boldsymbol{\delta}_{2^9}^{37}$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 0, p_5 = 1, p_6 = 1, \\ p_7 = 0, p_8 = 1, p_9 = 1. \end{cases}$$

10) $\boldsymbol{x}_{10} = \boldsymbol{\delta}_{2^9}^{39}$, or

$$\begin{cases} p_1 = 1, p_2 = 1, p_3 = 1, \\ p_4 = 0, p_5 = 1, p_6 = 1, \\ p_7 = 0, p_8 = 0, p_9 = 1. \end{cases}$$

Before ending this section we consider a general form of logic equations:

$$f(p_1,p_2,\ldots,p_n) = g(q_1,q_2,\ldots,q_m). \tag{45}$$

Consider a logic equation as Eq. (45), we want to find its algebraic form. To see why it is necessary to consider this kind of equations, we consider the following example. In Ref. [18], it is defined that a min-max expression is called an inseparable map if the logic equation

$$F(\boldsymbol{0},\boldsymbol{x}) = \boldsymbol{x}, \quad \boldsymbol{x} \in D^k, \tag{46}$$

has only solutions $[1 \ \cdots \ 1]$ or $[0 \ \cdots \ 0]$. It is obvious that Eq. (46) has the form of Eq. (45).

Proposition 9 The algebraic form of logic Eq. (45) is

$$\boldsymbol{M}_{\mathrm{e}}\boldsymbol{M}_{\mathrm{f}}(\boldsymbol{I}_{2^n}\otimes\boldsymbol{M}_{\mathrm{g}})p_1p_2\cdots p_nq_1q_2\cdots q_m = \boldsymbol{\delta}_2^1. \tag{47}$$

Proof Denote $p := f(p_1,p_2,\ldots,p_n)$ and $q := g(q_1,q_2,\ldots,q_m)$.

Equation (45) means either both p and q are "True" or both p and q are "False". That is, $p \leftrightarrow q = 1$. In algebraic form we have $\boldsymbol{M}_{\mathrm{e}}\boldsymbol{M}_{\mathrm{f}}p_1p_2\cdots p_{\mathrm{n}}\boldsymbol{M}_{\mathrm{g}}q_1q_2\cdots q_m = \boldsymbol{\delta}_2^1$.

Note that

$$p_1p_2\cdots p_n\boldsymbol{M}_{\mathrm{g}} = I_{2^n}\otimes\boldsymbol{M}_{\mathrm{g}}p_1p_2\cdots p_n.$$

Equation (47) follows immediately.

6 Logic inference

The purpose of the section is to deduct logic inference by solving logic equation. We will discuss it by examples.

Example 8 A said: "B is a liar", B said: "C is a liar", C said: "both A and B are liars". Who is the liar?

To solve this problem we define three logic variables as p : A is honest; q : B is honest; r : C is honest.

Then the three statements can be converted in logic version as

$$\begin{cases} p \Leftrightarrow \neg q, \\ q \Leftrightarrow \neg r, \\ r \Leftrightarrow [(\neg p) \wedge (\neg q)]. \end{cases} \tag{48}$$

Let $\boldsymbol{c} = \boldsymbol{\delta}_2^1$. Then Eq. (48) can be converted into algebraic form as

$$\begin{cases} \boldsymbol{M}_\mathrm{e} p \boldsymbol{M}_\mathrm{n} q = \boldsymbol{c}, \\ \boldsymbol{M}_\mathrm{e} q \boldsymbol{M}_\mathrm{n} r = \boldsymbol{c}, \\ \boldsymbol{M}_\mathrm{e} r \boldsymbol{M}_\mathrm{c} \boldsymbol{M}_\mathrm{n} p \boldsymbol{M}_\mathrm{n} q = \boldsymbol{c}. \end{cases} \tag{49}$$

It is easy to convert Eq. (49) into an algebraic form as

$$\boldsymbol{L}\boldsymbol{x} = \boldsymbol{b},$$

where $\boldsymbol{x} = pqr$, $\boldsymbol{b} = \boldsymbol{c}^3 = \boldsymbol{\delta}_8^1$.

It is easy to calculate that

$$\boldsymbol{L} = \boldsymbol{\delta}_8[8 \quad 5 \quad 3 \quad 2 \quad 4 \quad 1 \quad 5 \quad 8].$$

Since only $\boldsymbol{L}_6 = \boldsymbol{b}$, we have unique solution $\boldsymbol{x} = \boldsymbol{\delta}_8^6$, which implies that $p=0$, $q=1$, $r=0$.

It was said that only B is honest.

Example 9 A competition of five players is going on a simple-rotating way, which means each player has to play with all the others. We have the following information about the result:

1) C beat E;
2) A won 3 games;
3) E won 1 game;
4) Among B, C, D, there is one player, who beat the other two;
5) Each of B, C, D won 2 games;
6) Each of A, C, D, E won some and lost some.

Using AB to denote "A beat B" and so on, it is clear from the definition that

$$BA = \neg AB, \ CA = \neg AC, \ldots$$

Next, each statement is converted into a logic expression.

1) C beat E

$$CE = 1.$$

2) A won 3 games

$$\begin{cases} (AB \wedge AC \wedge AD) \vee (AB \wedge AC \wedge AE) \\ \quad \vee (AB \wedge AD \wedge AE) \vee (AC \wedge AD \wedge AE) = 1, \\ AB \wedge AC \wedge AD \wedge AE = 0. \end{cases} \tag{50}$$

3) E won 1 game

$$\begin{cases} AE \wedge BE \wedge DE = 0, \\ (EA \wedge EB) \vee (EA \wedge EC) \vee (EA \wedge ED) \\ \quad \vee (EB \wedge ED) \vee (EC \wedge ED) = 0. \end{cases}$$

Since $EC = \neg CE = 0$, it can be removed from the above expression to simplify it to be

$$\begin{cases} AE \wedge BE \wedge DE = 0, \\ (EA \wedge EB) \vee (EA \wedge ED) \vee (EB \wedge ED) = 0. \end{cases} \tag{51}$$

4) Among B, C, D, one player beat the other two

$$(BC \wedge BD) \vee (CB \wedge CD) \vee (DB \wedge DC) = 1. \tag{52}$$

5) Each of B, C, D won 2 games

i) B won 2 games:

$$\begin{cases} (BA \wedge BC) \vee (BA \wedge BD) \vee (BA \wedge BE) \\ \quad \vee (BC \wedge BD) \vee (BC \wedge BE) \vee (BD \wedge BE) = 1, \\ (BA \wedge BC \wedge BD) \vee (BA \wedge BC \wedge BE) \\ \quad \vee (BA \wedge BD \wedge BE) \vee (BC \wedge BD \wedge BE) = 0. \end{cases} \tag{53}$$

ii) C won 2 games: Note that $CE=1$ can be used to simplify the expression. Then we have

$$\begin{cases} CA \vee CB \vee CD = 1, \\ (CA \wedge CB) \vee (CA \wedge CD) \vee (CB \wedge CD) = 0. \end{cases} \tag{54}$$

iii) D won 2 games:

$$\begin{cases} (DA \wedge DB) \vee (DA \wedge DC) \vee (DA \wedge DE) \\ \quad \vee (DB \wedge DC) \vee (DB \wedge DE) \vee (DC \wedge DE) = 1, \\ (DA \wedge DB \wedge DC) \vee (DA \wedge DB \wedge DE) \\ \quad \vee (DA \wedge DC \wedge DE) \vee (DB \wedge DC \wedge DE) = 0. \end{cases} \tag{55}$$

6) Each of A, C, D, E won some and lost some Obviously, this statement does not contain additional information.

Next, Eqs. (50)–(55) are converted into algebraic form. To save space, denote

$$p = AB, \ q = AC, \ r = AD, \ s = AE, \ t = BC,$$
$$u = BD, \ v = BE, \ \alpha = CD, \ \beta = DE.$$

Using De Morgan's law to the second equation of Eq. (51) and equations in Eq. (55), and putting all the algebraic equations together yield

$$\begin{aligned}
&\boldsymbol{M}_{\mathrm{d}}^{3}\boldsymbol{M}_{\mathrm{c}}^{2}pqr\boldsymbol{M}_{\mathrm{c}}^{2}pqs\boldsymbol{M}_{\mathrm{c}}^{2}prs\boldsymbol{M}_{\mathrm{c}}^{2}qrs = \boldsymbol{v}_{\mathrm{T}},\\
&\boldsymbol{M}_{\mathrm{c}}^{3}pqrs = \boldsymbol{v}_{\mathrm{F}},\\
&\boldsymbol{M}_{\mathrm{c}}^{2}sv\beta = \boldsymbol{v}_{\mathrm{F}},\\
&\boldsymbol{M}_{\mathrm{c}}^{2}\boldsymbol{M}_{\mathrm{d}}sv\boldsymbol{M}_{\mathrm{d}}s\beta\boldsymbol{M}_{\mathrm{d}}v\beta = \boldsymbol{v}_{\mathrm{T}},\\
&\boldsymbol{M}_{\mathrm{d}}^{2}\boldsymbol{M}_{\mathrm{c}}tu\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{n}}t\alpha\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{n}}u\boldsymbol{M}_{\mathrm{n}}\alpha = \boldsymbol{v}_{\mathrm{T}},\\
&\boldsymbol{M}_{\mathrm{d}}^{5}\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{n}}pt\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{n}}pu\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{n}}pv\boldsymbol{M}_{\mathrm{c}}tuv\boldsymbol{M}_{\mathrm{c}}tv\boldsymbol{M}_{\mathrm{c}}uv = \boldsymbol{v}_{\mathrm{T}},\\
&\boldsymbol{M}_{\mathrm{d}}^{3}\boldsymbol{M}_{\mathrm{c}}^{2}\boldsymbol{M}_{\mathrm{n}}ptu\boldsymbol{M}_{\mathrm{c}}^{2}\boldsymbol{M}_{\mathrm{n}}ptv\boldsymbol{M}_{\mathrm{c}}^{2}\boldsymbol{M}_{\mathrm{n}}puv\boldsymbol{M}_{\mathrm{c}}^{2}tuv = \boldsymbol{v}_{\mathrm{F}},\\
&\boldsymbol{M}_{\mathrm{d}}^{2}\boldsymbol{M}_{\mathrm{n}}q\boldsymbol{M}_{\mathrm{n}}t\alpha = \boldsymbol{v}_{\mathrm{T}},\\
&\boldsymbol{M}_{\mathrm{d}}^{2}\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{n}}q\boldsymbol{M}_{\mathrm{n}}t\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{n}}q\alpha\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}_{\mathrm{n}}t\alpha = \boldsymbol{v}_{\mathrm{F}},\\
&\boldsymbol{M}_{\mathrm{c}}^{5}\boldsymbol{M}_{\mathrm{d}}ru\boldsymbol{M}_{\mathrm{d}}r\alpha\boldsymbol{M}_{\mathrm{d}}r\boldsymbol{M}_{\mathrm{n}}\beta\boldsymbol{M}_{\mathrm{d}}u\alpha\boldsymbol{M}_{\mathrm{d}}u\boldsymbol{M}_{\mathrm{n}}\beta\boldsymbol{M}_{\mathrm{d}}\alpha v\boldsymbol{M}_{\mathrm{n}}\beta = \boldsymbol{v}_{\mathrm{F}},\\
&\boldsymbol{M}_{\mathrm{c}}^{3}v\boldsymbol{M}_{\mathrm{d}}^{2}ru\alpha\boldsymbol{M}_{\mathrm{d}}^{2}ru\boldsymbol{M}_{\mathrm{n}}\beta\boldsymbol{M}_{\mathrm{d}}^{2}r\alpha\boldsymbol{M}_{\mathrm{n}}\beta\boldsymbol{M}_{\mathrm{d}}^{2}u\alpha\boldsymbol{M}_{\mathrm{n}}\beta = \boldsymbol{v}_{\mathrm{T}},
\end{aligned} \tag{56}$$

where

$$\boldsymbol{v}_{\mathrm{F}} = \boldsymbol{\delta}_2^2,\ \boldsymbol{v}_{\mathrm{T}} = \boldsymbol{\delta}_2^1.$$

Now multiplying all the equations in Eq. (56) together and using the standard procedure, we have an algebraic form as

$$\boldsymbol{L}\boldsymbol{x} = \boldsymbol{b}, \tag{57}$$

where $\boldsymbol{x} = pqrstuv\alpha\beta$. Using Eq. (31) yields

$$\boldsymbol{b} = \boldsymbol{v}_{\mathrm{T}}\boldsymbol{v}_{\mathrm{F}}^2\boldsymbol{v}_{\mathrm{T}}^3\boldsymbol{v}_{\mathrm{F}}\boldsymbol{v}_{\mathrm{T}}\boldsymbol{v}_{\mathrm{F}}^2\boldsymbol{v}_{\mathrm{T}} = \boldsymbol{\delta}_{2^{11}}^{791}.$$

$\boldsymbol{L}$ is a $2^{11}\times 2^9$ matrix. The first and last few columns are

$$\begin{aligned}\boldsymbol{\delta}_{2^{11}}[&5 \quad 261 \quad 15 \quad 269 \quad 277 \quad 405 \quad 287 \quad 413\\ &\cdots\\ &1812 \quad 1939 \quad 1812 \quad 1940 \quad 1972 \quad 1971 \quad 1972 \quad 1972].\end{aligned}$$

A routine shows that

$$\boldsymbol{L}_{69} = \boldsymbol{L}_{135} = \boldsymbol{L}_{140} = \boldsymbol{L}_{284} = \boldsymbol{b}.$$

So the solutions of Eqs. (50)–(55) are

$$\boldsymbol{x}_1 = \boldsymbol{\delta}_{2^9}^{69},\ \boldsymbol{x}_2 = \boldsymbol{\delta}_{2^9}^{135},\ \boldsymbol{x}_3 = \boldsymbol{\delta}_{2^9}^{140},\ \boldsymbol{x}_4 = \boldsymbol{\delta}_{2^9}^{284}. \tag{58}$$

Using Eq. (30) yields Table 2.

Table 2 Solutions of Eqs. (50)–(55)

	p	q	r	s	t	u	v	α	β
$\boldsymbol{x}_1$	1	1	0	1	1	1	0	1	1
$\boldsymbol{x}_2$	1	0	1	1	1	1	0	0	1
$\boldsymbol{x}_3$	1	0	1	1	1	0	1	0	0
$\boldsymbol{x}_4$	0	1	1	1	0	0	1	0	0

Next, we modify the last statement "Each of A, C, D, E won some and lost some" to the following:

Among the group A, C, D, E, each won some and lost some.

Now it is obvious that the new information is: A cannot win all C, D, E; E cannot lose to all A, C, D, (equivalently to A and D). All other parts of information have already been implied by previous statements. Then we have two more equations:

$$\begin{cases} q \wedge r \wedge s = 0,\\ s \wedge \beta = 0. \end{cases} \tag{59}$$

Equivalently, we have algebraic equations as

$$\begin{cases} \boldsymbol{M}_{\mathrm{c}}^2 qrs = \boldsymbol{v}_{\mathrm{F}},\\ \boldsymbol{M}_{\mathrm{c}} s\beta = \boldsymbol{v}_{\mathrm{F}}. \end{cases} \tag{60}$$

One way to solve this problem is to add Eq. (60) to Eq. (56) and solve this system of equations again. Obviously, this is a heavy job. From Table 2, it is easy to check that only $\boldsymbol{x}_3$ satisfies Eq. (60). So in this case $\boldsymbol{x}_3$ is the unique solution.

7 Conclusions

Logic is an important tool in control. However, there are few general tools to deal with logical problems. In this paper, the semi-tensor product of matrixes was introduced. Then it was used to express logic equations. Under this framework, a logic equation can be converted into a linear algebraic equation. Then it is easily solvable. Some interesting examples were included to demonstrate that this new approach is applicable to logic inference.

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