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Automatica 44 (2008) 1996–2005
automatica
www.elsevier.com/locate/automatica

Constructive stabilization for quadratic input nonlinear systems☆

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Received 9 November 2006; received in revised form 29 October 2007; accepted 1 January 2008
Available online 4 March 2008

Abstract

In this paper stabilization of nonlinear systems with quadratic multi-input is considered. With the help of control Lyapunov function (CLF), a constructive parameterization of controls that globally asymptotically stabilize the system is proposed. Two different cases are considered. Firstly, under certain regularity assumptions, the feasible control set is parameterized, and continuous feedback stabilizing controls are designed. Then for the general case, piecewise continuous stabilizing controls are proposed. The design procedure can also be used to verify whether a candidate CLF is indeed a CLF. Several illustrative examples are presented as well.

Keywords: Quadratic input nonlinear system; Global stabilization; Control Lyapunov function

1. Introduction

Stabilization of control systems is one of the most important topics in control theory. The most useful tool in stability analysis and stabilizing control design is the Lyapunov approach, although in practice it is sometimes difficult to construct a Lyapunov function. When stabilization is considered, the selection of a candidate Lyapunov function has to be considered simultaneously with the design of control. The notion of control Lyapunov function (CLF) introduced by Artstein gives a way to consider the choice of Lyapunov function and the design of control simultaneously (Artstein, 1983). Under some mild constraints on the feasible control, Artstein pointed out that for a class of nonlinear control systems, the stabilizability is equivalent to the existence of a CLF in Artstein (1983).

Many results concerning CLFs have been obtained in the literature. For example, Tsinias gave some sufficient conditions for the existence of CLFs for affine nonlinear control systems with some special forms, and provided the explicit constructions of CLFs (Tsinias, 1990, 1991). Faubourg and Pomet designed the explicit CLFs for affine homogeneous systems without drift satisfying the Jurdejevic–Quinn conditions (Faubourg & Pomet, 2000). Recently, Mazence and Malisoff constructed CLFs for non-affine nonlinear control systems satisfying the Jurdejevic–Quinn conditions (Mazenc & Malisoff, 2006).

Designing stabilizing controls constructively from a known CLF is in itself also an interesting and important topic. For affine nonlinear control systems, Sontag gave a universal construction of the state feedback control law via a known CLF (Sontag, 1989). More recently, Curtis et al. presented a constructive parameterization of universal formulas of the state feedback control law with respect to a given CLF, and proved that parameterization is complete in Curtis and Beard (2004) using the introduced notion in the satisficing decision theory (Goodrich, Stirling, & Frost, 1998; Srirling, 2003; Srirling & Morrell, 1991).

In general there may not exist a continuous stabilizing control for non-affine nonlinear control systems. Recently, Moulay and Perruquetti provided a sufficient condition for the existence of a continuous stabilizing control for the non-affine (quadratic input) nonlinear control system: the existence of a CLF satisfying the small control property, and a convexity property with respect to the control input (Moulay

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Zhihua Qu under the direction of Editor Hassan Khalil. This work is supported partly by NNSF of China under Grants 60221301 and 60334040, partly by Sida-VR Swedish Research Links Grant 348-2006-5894.
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0005-1098/$ - see front matter © 2008 Published by Elsevier Ltd
doi:10.1016/j.automatica.2008.01.005

& Perruquetti, 2005). However, their proof is non-constructive, except for special case of polynomial single input of degree two and three.

Nowadays, quadratic input nonlinear systems appear in many practical fields, such as in magnetic problems (Moulay & Perruquetti, 2005) and oscillation problems of the electromagnetic oscillator (Sane & Bernstein, 2002). Therefore, the investigation of such systems is practically relevant.

In this paper, we consider the following quadratic input system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i + \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} h_{i_1 i_2}(x)u_{i_1}u_{i_2}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and f, g, h_{ij} are the smooth vector fields with $f(0) = 0$.

Moulay and Perruquetti gave a method of constructing feedback control laws for system (1) with $m = 1$, and the case when $h_{ij}(x) \equiv 0, i \neq j$, which is very similar to the case of $m = 1$ (Moulay & Perruquetti, 2005). We note that it has been pointed out in Moulay and Perruquetti (2005) that the general multi-input case is much more difficult. Lin derived an arbitrarily small state feedback control law which globally asymptotically stabilizes system (1) (Lin, 1995, 1996). However, one of the assumptions in Lin (1995, 1996) is that there exists a C^r $(r \geq 1)$ function $V : \mathbb{R}^n \to \mathbb{R}$, which is positive definite and proper on $\mathbb{R}^n$ such that the unforced dynamic system

$$\dot{x} = f(x) \tag{2}$$

is Lyapunov stable, i.e., $L_f V(x) \leq 0, \forall x \in \mathbb{R}^n$.

In this paper we study the problem of designing constructively a globally stabilizing control for system (1) via a known CLF. In particular, we give a parameterization of the continuous stabilizing feedback controls for system (1) without assuming that system (2) is Lyapunov stable. Moreover, we show that the parameterization is complete.

The paper is organized as follows. Section 2 presents some preliminaries. Section 3 investigates the feasible set of stabilizing controls. Section 4 provides a detailed control design procedure with certain assumptions. Section 5 investigates the assumptions used in Sections 3 and 4 and shows that all the assumptions lead to the regularity assumption. Section 6 considers the singular case and the design of piecewise continuous stabilizing controls. Section 7 gives some concluding remarks, and some discussion on further applications and investigations.

2. Preliminaries

In this section we first recall some basic definitions and facts concerning control Lyapunov functions, then we give some notations.

Consider a non-affine nonlinear control system

$$\dot{x} = f(x, u), \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and f is the smooth vector field with $f(0, 0) = 0$.

Definition 1 (*Moulay & Perruquetti, 2005*). A smooth, proper, and positive definite function V is a control Lyapunov function (CLF) for system (3), if for any $x \in \mathbb{R}^n \setminus \{0\}$

$$\inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x} f(x, u) \right\} < 0. \tag{4}$$

Definition 2 (*Moulay & Perruquetti, 2005*). A CLF V for system (3) is said to satisfy the small control property, if for each $\epsilon > 0$ there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some u with $\|u\| < \epsilon$ such that

$$\frac{\partial V}{\partial x} f(x, u) < 0. \tag{5}$$

From the above definition, one can see that the small control property assures the existence of a stabilizing control u which is continuous at the origin with $u(0) = 0$.

The control design based on the CLF is due to Artstein (1983) and revisited with a simplified proof in Moulay and Perruquetti (2005).

Proposition 3 (*Artstein, 1983; Moulay & Perruquetti, 2005*). *If there exists a CLF V for system (3) such that $u \to \frac{\partial V}{\partial x} f(x, u)$ is convex for all $x \in \mathbb{R}^n \setminus \{0\}$, then system (3) is globally asymptotically stabilizable by a state feedback control law $u = u(x)$ with $u(0) = 0$ that is continuous over $\mathbb{R}^n \setminus \{0\}$. In addition, if the CLF V satisfies the small control property, then $u(x)$ is continuous over $\mathbb{R}^n$.*

However, the construction of a stabilizing control is in general highly nontrivial, with the exception of the affine control case. For affine nonlinear control systems Sontag provided a formula to construct a continuous stabilizer via a known Lyapunov function (Sontag, 1989).

Throughout this paper, we denote $\mathcal{B} = \{\varsigma \in \mathbb{R}^m | \|\varsigma\| < 1\}$. We also denote A^+ for pseudo-inverse of the matrix A (Penrose, 1995).

3. Feasible set of controls

Consider system (1). Denote by $h^i_{i_1 i_2}(x)$ the ith component of $h_{i_1 i_2}(x)$, and for each $i \in \{1, \ldots, n\}$, let

$$R_i(x) = \begin{bmatrix} h^i_{11}(x) & \frac{h^i_{12}(x) + h^i_{21}(x)}{2} & \cdots & \frac{h^i_{1m}(x) + h^i_{m1}(x)}{2} \\ \frac{h^i_{12}(x) + h^i_{21}(x)}{2} & h^i_{22}(x) & \cdots & \frac{h^i_{2m}(x) + h^i_{m2}(x)}{2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{h^i_{1m}(x) + h^i_{m1}(x)}{2} & \frac{h^i_{2m}(x) + h^i_{m2}(x)}{2} & \cdots & h^i_{mm}(x) \end{bmatrix}.$$

Then

$$\sum_{i_1=1}^{m}\sum_{i_2=1}^{m} h_{i_1 i_2}(x)u_{i_1}u_{i_2} = [u^{\mathrm{T}} R_1(x) u, \ldots, u^{\mathrm{T}} R_n(x) u]^{\mathrm{T}}.$$

Throughout this paper, we assume V is a CLF for system (1). Then the time derivative of V along the trajectories of system (1) is

$$\dot{V}|_{(1)} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u + u^{\mathrm{T}} \left(\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} R_i(x) \right) u.$$

Let $a(x) = \frac{\partial V}{\partial x} f(x)$, $b(x) = \left(\frac{\partial V}{\partial x} g(x)\right)^{\mathrm{T}}$, and $R(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} R_i(x)$. Then $R(x)$ is symmetric and

$$\dot{V}|_{(1)} = a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}}R(x)u. \tag{6}$$

Denote

$$F(x,u) = a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}}R(x)u.$$

Lemma 4. *Assume V is a CLF for system (1). Then the following two conditions are equivalent.*

1. *$F(x,u)$ is (strictly) convex with respect to u for all $x \neq 0$.*
2. *$R(x) \geq 0$ (respectively, $R(x) > 0$) for all $x \neq 0$.*

Based on the discussion, it is reasonable that we emphasize on the case where $F(x,u)$ is convex or equivalently, $R(x)$ is positive semi-definite. We thus assume first

Assumption 5. $F(x,u)$ is strictly convex with respect to u for all $x \neq 0$.

According to Lemma 4, Assumption 5 is equivalent to $R(x) > 0, \forall x \neq 0$.

Assumption 5 is stronger than the corresponding assumption in Proposition 3, where $F(x,u)$ is only convex with respect to u for all $x \neq 0$. We will relax Assumption 5 later.

Lemma 6 (*Curtis & Beard, 2004*). *Assume A is a symmetric positive definite matrix, then the set of solutions to the quadratic inequality*

$$\xi^{\mathrm{T}}A\xi + d^{\mathrm{T}}\xi + c < 0, \quad \xi \in \mathbb{R}^m$$

is nonempty if and only if

$$\frac{1}{4}d^{\mathrm{T}}A^{-1}d - c > 0,$$

and the set of solutions is given by

$$\xi = -\frac{1}{2}A^{-1}d + A^{-\frac{1}{2}}\nu\sqrt{\frac{1}{4}d^{\mathrm{T}}A^{-1}d - c}, \quad \nu \in \mathcal{B}.$$

Note that $R(x)$ is symmetric and supposed positive definite except at the origin. Then we have the following result:

Corollary 7. *Assume V is a CLF for system (1), and Assumption 5 holds. Then a stabilizing control $u_0 = u_0(x)$ of system (1) can be expressed as*

$$\begin{aligned} u_0(x) &= -\frac{1}{2}R^{-1}(x)b(x) \\ &\quad + R^{-\frac{1}{2}}(x)\nu(x)\sqrt{\frac{1}{4}b^{\mathrm{T}}(x)R^{-1}(x)b(x) - a(x)}, \\ &\quad \nu(x) \in \mathcal{B}, \quad \forall x \neq 0, \end{aligned} \tag{7}$$

and the set of continuous (over $\mathbb{R}^n \setminus \{0\}$) stabilizing controls $u_0(x)$ is parameterized by (7) with any continuous (over $\mathbb{R}^n \setminus \{0\}$) function $\nu(x) \in \mathcal{B}$. Moreover, if the CLF V for system (1) satisfies the small control property, then $u_0(x)$ is continuous over $\mathbb{R}^n$ with $u_0(0) = 0$.

Next, we consider a more general case, where $R(x) \geq 0$. Equivalently, we assume:

Assumption 8. $F(x,u)$ is convex with respect to u for all $x \neq 0$.

In the following we construct the set of stabilizing controls under the assumptions that V is a CLF for system (1) and Assumption 8 holds.

Assume V is a CLF for system (1), we define

$$\xi(x) := \inf_{u \in \mathbb{R}^m} [a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}}R(x)u].$$

Then $\xi(x) \in [-\infty, 0)$, $\forall x \neq 0$, and $\xi(0) = 0$, since $a(0) = 0$, $b(0) = 0$, and $R(0) = 0$. For any $\alpha < 0$, we define a truncated ξ as $\xi_\alpha(x) = \max\{\xi(x), \alpha\}$. It is easy to see that if $\xi(x)$ is continuous, then $\xi_\alpha(x)$ is also continuous.

Now for each fixed x, we decompose u as

$$u(x) = u_I(x) + u_P(x), \tag{8}$$

where

$$\begin{aligned} u_I(x) &= R(x)R^{+}(x)u \in \mathrm{Im}(R(x)), \\ u_P(x) &= (I - R(x)R^{+}(x))u \in \mathrm{Im}^{\perp}(R(x)). \end{aligned}$$

Note that $R(x)$ is symmetric, so

$$R(x)R^{+}(x) = (R^{+}(x)R(x))^{\mathrm{T}} = R^{+}(x)R(x).$$

Then

$$\begin{aligned} &a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}}R(x)u \\ &\quad = a(x) + b^{\mathrm{T}}(x)u_I + b^{\mathrm{T}}(x)u_P + u_I^{\mathrm{T}}R(x)u_I. \end{aligned} \tag{9}$$

Assume V is a CLF for system (1), we also define

$$\eta(x) := \inf_{u_I \in \mathrm{Im}(R(x))} [a(x) + b^{\mathrm{T}}(x)u_I + u_I^{\mathrm{T}}R(x)u_I]. \tag{10}$$

Note that $\eta(x) > -\infty$ is a well-posed function.

Denote

$$\mathcal{D} = \{ f \in C_0(\mathbb{R}^n \setminus \{0\}) \big| 0 < f(x) < 1\},$$

i.e., $\mathcal{D}$ is the set of continuous functions on $\mathbb{R}^n \setminus \{0\}$ with their values in (0, 1). Choosing an $\alpha < 0$ and a $\mu(x) \in \mathcal{D}$, along with (9), we have

$$\begin{aligned} &\inf_{u \in \mathbb{R}^m} [a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}}R(x)u] \\ &\quad = \inf_{u_I \in \mathrm{Im}(R(x))} [a(x) + b^{\mathrm{T}}(x)u_I + u_I^{\mathrm{T}}R(x)u_I - \eta(x) \\ &\qquad + \mu(x)\xi_\alpha(x)] \\ &\qquad + \inf_{u_P \in \mathrm{Im}^{\perp}(R(x))} [\eta(x) - \mu(x)\xi_\alpha(x) + b^{\mathrm{T}}(x)u_P]. \end{aligned} \tag{11}$$

For the first term of the right-hand side of (11), we have

$$\begin{aligned} &\inf_{u_I \in \mathrm{Im}(R(x))} [a(x) + b^{\mathrm{T}}(x)u_I + u_I^{\mathrm{T}}R(x)u_I - \eta(x) \\ &\qquad + \mu(x)\xi_\alpha(x)] \\ &\quad = \mu(x)\xi_\alpha(x) < 0, \quad \forall x \neq 0. \end{aligned} \tag{12}$$

For the second term, we have

$$\inf_{u_P\in \mathrm{Im}^{\perp}(R(x))} [\eta(x) - \mu(x)\xi_\alpha(x) + b^{\mathrm{T}}(x)u_P] = \xi(x) - \mu(x)\xi_\alpha(x) \le (1-\mu(x))\xi_\alpha(x) < 0, \quad \forall x \ne 0. \tag{13}$$

Next, we claim that we can construct continuous controls u_I and u_P separately. Then $u = u_I + u_P$ is a feasible control. Set

$$a_1(x) = a(x) - \eta(x) + \mu(x)\xi_\alpha(x),$$
$$a_2(x) = \eta(x) - \mu(x)\xi_\alpha(x).$$

From (12) and (13) we have the following result:

Lemma 9. *Assume V is a CLF for system (1). Then there exist $u_I \in \mathrm{Im}(R(x))$ and $u_P \in \mathrm{Im}^{\perp}(R(x))$, such that*

$$a_1(x) + b^{\mathrm{T}}(x)u_I + u_I^{\mathrm{T}} R(x) u_I < 0, \quad \forall x \ne 0, \tag{14}$$

and

$$a_2(x) + b^{\mathrm{T}}(x)u_P < 0, \quad \forall x \ne 0. \tag{15}$$

Remark 10. The above lemma suggests that: as long as

$$\inf_{u\in\mathbb{R}^m} [a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}}R(x)u] < 0, \quad \forall x \ne 0, \tag{16}$$

we have

$$\inf_{u_I\in \mathrm{Im}(R(x))} [a_1(x) + b^{\mathrm{T}}(x)u_I + u_I^{\mathrm{T}}R(x)u_I] < 0, \quad \forall x \ne 0, \tag{17}$$

and

$$\inf_{u_P\in \mathrm{Im}^{\perp}(R(x))} [a_2(x) + b^{\mathrm{T}}(x)u_P] < 0, \quad \forall x \ne 0. \tag{18}$$

In preparation for the construction of $u(x)$, we also need the following lemma, which can be proved easily.

Lemma 11. *If $u_I(x)$ is a continuous (except at $x = 0$) control satisfying (14) and $u_P(x)$ is a continuous (except at $x = 0$) control satisfying (15), then $u(x) = u_I(x) + u_P(x)$ is continuous (except at $x = 0$), and satisfies*

$$a(x) + b^{\mathrm{T}}(x)u(x) + u(x)^{\mathrm{T}}R(x)u(x) < 0, \quad \forall x \ne 0. \tag{19}$$

Finally, we construct three feasible sets as:

$$\Phi_\mu^\alpha = \left\{ u_I(x) \in \mathrm{Im}(R(x)) \,\middle|\, a_1(x) + b^{\mathrm{T}}(x)u_I(x) + u_I^{\mathrm{T}}(x)R(x)u_I(x) < 0, \forall x \ne 0;\, u_I(0) = 0 \right\},$$

$$\Psi_\mu^\alpha = \left\{ u_P(x) \in \mathrm{Im}^{\perp}(R(x)) \,|\, a_2(x) + b^{\mathrm{T}}(x)u_P(x) < 0, \forall x \ne 0;\, u_P(0) = 0 \right\},$$

$$\Phi = \left\{ u(x) \in \mathbb{R}^m \,\middle|\, a(x) + b^{\mathrm{T}}(x)u(x) + u^{\mathrm{T}}(x)R(x)u(x) < 0, \forall x \ne 0;\, u(0) = 0 \right\}.$$

Based on the above arguments, after some tedious but straightforward calculations that are skipped here, we obtain the first main result as follows. It provides a complete parameterized expression of the feasible set of u, from which stabilizing continuous controls can be constructed.

Theorem 12. *Assume V is a CLF for system (1), and Assumption 8 holds. Then the feasible set of stabilizing controls u for system (1) consists of*

$$\Phi = \bigcup_{\alpha<0} \bigcup_{\mu(x)\in\mathcal{D}} \{u = u_I(x) + u_P(x) | u_I(x) \in \Phi_\mu^\alpha, u_P(x) \in \Psi_\mu^\alpha\}. \tag{20}$$

Proof. According to Lemmas 9 and 11, Φ is composed of double commutative unions (the parameters α and $\mu(x)$ are independent). □

4. Control design

In this section, we construct continuous $u_I(x)$ from (17) and continuous $u_P(x)$ from (18).

For the sake of presentation, we tentatively assume

Assumption 13. 1. For any $\alpha < 0$, $\xi_\alpha(x)$ is continuous over $\mathbb{R}^n \setminus \{0\}$;
2. $\eta(x)$ is continuous over $\mathbb{R}^n \setminus \{0\}$.

We consider (17) first. It is similar to the case of $R(x) > 0$, $\forall x \ne 0$. However, due to the restriction on u_I, certain further investigation has to be carried out. We will reduce the expression to the standard case, i.e., $R(x) > 0$, $\forall x \ne 0$.

Lemma 14. *Let A be an $n \times n$ symmetric matrix, $u \in \mathbb{R}^n$ and $v = Au$. Then $u = A^+v$ if and only if $u \in \mathrm{Im}(A)$.*

We need one more assumption in order to construct continuous controls.

Assumption 15. $R^+(x)$ (or equivalently $(R^{\frac{1}{2}})^+(x)$) is continuous over $\mathbb{R}^n \setminus \{0\}$.

Observe that if we want u_I and u_P to be continuous, then the assumptions in Assumption 13 become necessary. However, we should first check whether the assumptions in Assumption 13 are reasonable. In the following, we give a sufficient condition of the second assumption in Assumption 13.

Lemma 16. *If V is a CLF for system (1), and Assumptions 8 and 15 hold, then $\eta(x)$ is continuous over $\mathbb{R}^n \setminus \{0\}$.*

Proof. Recall that

$$\eta(x) = \inf_{u_I\in \mathrm{Im}(R(x))} [a(x) + b^{\mathrm{T}}(x)u_I + u_I^{\mathrm{T}}R(x)u_I]. \tag{21}$$

Let $v = R^{\frac{1}{2}}(x)u_I$. By Lemma 14, $u_I = (R^{\frac{1}{2}})^+(x)v$. Hence

$$\begin{aligned} &a(x) + b^{\mathrm{T}}(x)u_I + u_I^{\mathrm{T}}R(x)u_I \\ &= a(x) + b^{\mathrm{T}}(x)(R^{\frac{1}{2}})^+(x)v + v^{\mathrm{T}}v \\ &= \left[v + \frac{1}{2}(R^{\frac{1}{2}})^+(x)b(x)\right]^{\mathrm{T}} \left[v + \frac{1}{2}(R^{\frac{1}{2}})^+(x)b(x)\right] \\ &\quad + a(x) - \frac{1}{4}b^{\mathrm{T}}(x)R^+(x)b(x). \end{aligned}$$

It is clear that

$$\eta(x) = a(x) - \frac{1}{4} b^{\mathrm{T}}(x) R^{+}(x) b(x).$$

It follows that if $R^{+}(x)$ is continuous over $\mathbb{R}^n \setminus \{0\}$, then so is $\eta(x)$. $\square$

Next, we consider $\xi(x)$. Assume V is a CLF for system (1) and Assumption 8 holds. Note that

$$\xi(x) = \inf_{u \in \mathbb{R}^m} [a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}} R(x) u]. \tag{22}$$

Split $b(x)$ as $b(x) = b_I(x) + b_P(x)$, where $b_I \in \mathrm{Im}(R(x))$, and $b_P \in \mathrm{Im}^{\perp}(R(x))$. Then we have

$$\begin{aligned} &a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}} R(x) u \\ &\quad = \left[v + \frac{1}{2}(R^{\frac{1}{2}})^{+}(x) b(x) \right]^{\mathrm{T}} \left[v + \frac{1}{2}(R^{\frac{1}{2}})^{+}(x) b(x) \right] \\ &\qquad + a(x) - \frac{1}{4} b^{\mathrm{T}}(x) R^{+}(x) b(x) + b_P^{\mathrm{T}}(x) u_P. \end{aligned}$$

Case 1: If $b_P(x) = 0$, i.e., $b(x) = b_I(x) \in \mathrm{Im}(R(x))$, then $b_P^{\mathrm{T}}(x) u_P = 0$. Hence $\xi(x) = \eta(x)$.

Case 2: If $b_P(x) \neq 0$, then we can choose $u_P = -d(x) b_P(x) \in \mathrm{Im}^{\perp}(R(x))$ with some positive function $d(x)$. It follows that $b_P^{\mathrm{T}}(x) u_P = -d(x)\|b_P(x)\|^2$. Since we can choose $d(x) > 0$ as large as we wish, it follows that $\xi(x) = -\infty$.

Combining the above two cases yields

$$\xi(x) = \begin{cases} -\infty, & \text{if } b(x) \notin \mathrm{Im}(R(x)) (\text{i.e., } b_P(x) \neq 0), \\ \eta(x), & \text{if } b(x) \in \mathrm{Im}(R(x)) (\text{i.e., } b_P(x) = 0). \end{cases} \tag{23}$$

We conclude that $\xi(x)$ (or even $\xi_\alpha(x)$) can be made continuous, if and only if $b_P(x) \equiv 0$ and $\eta(x)$ is continuous.

In fact, the case $b_P(x) \equiv 0$ is of less interest. It is equivalent to the case when $R(x)$ is positive definite except at the origin. Then in order to incorporate the case $b_P(x)$ is not identically zero, the only way is to modify $\xi_\alpha(x)$. Denote

$$\mathcal{S} = \{x \in \mathbb{R}^n \setminus \{0\} | b_P(x) = 0\}.$$

Then the distance $d(x, \mathcal{S})$ from x to $\mathcal{S}$ is a well-defined continuous function over $\mathbb{R}^n \setminus \{0\}$.

For any $\epsilon > 0$ and any $x \in \mathbb{R}^n \setminus \{0\}$, we can define a function as

$$\varphi^{\epsilon}(x) = \begin{cases} 1 - \dfrac{d(x, \mathcal{S})}{\epsilon}, & d(x, \mathcal{S}) \leq \epsilon, \\ 0, & d(x, \mathcal{S}) > \epsilon. \end{cases} \tag{24}$$

It is obviously continuous over $\mathbb{R}^n \setminus \{0\}$. Using it, we define

$$\xi_\alpha^{\epsilon}(x) = -\varphi^{\epsilon}(x)|\eta(x)| + (1 - \varphi^{\epsilon}(x))\alpha, \quad \forall x \neq 0. \tag{25}$$

Lemma 17. *Assume V is a CLF for system (1), and item 2 of Assumption 13 holds. Then $\xi_\alpha^{\epsilon}(x)$ is a continuous function over $\mathbb{R}^n \setminus \{0\}$. Moreover,*

$$\xi(x) \leq \xi_\alpha^{\epsilon}(x) < 0, \quad \forall x \neq 0. \tag{26}$$

Proof. Continuity is obvious. In the following, we prove Inequality (26) in two cases.

1. If $x \in \mathcal{S}$, then $\xi_\alpha^{\epsilon}(x) = -|\eta(x)| = -|\xi(x)| = \xi(x) < 0$. So Inequality (26) holds.
2. If $x \notin \mathcal{S}$ and $x \neq 0$, then $\xi(x) = -\infty$, thus the first inequality in (26) is trivially true. In the following, we show that the second inequality in (26) holds. First, note that $\mathcal{S}$ is a closed set in $\mathbb{R}^n \setminus \{0\}$. Then $d(x, \mathcal{S}) > 0, \forall\, 0 \neq x \notin \mathcal{S}$.
   (a) If $0 < d(x, \mathcal{S}) \leq \epsilon$, then the second inequality in (26) is valid because $-|\eta(x)|$ is non-positive and α is negative.
   (b) If $d(x, \mathcal{S}) > \epsilon$, then $\xi_\alpha^{\epsilon}(x) = \alpha < 0$. $\square$

Remark 18. 1. It is obvious that as $\epsilon \to 0_+$, $\xi_\alpha^{\epsilon}(x) \to \xi_\alpha(x)$.

2. Hereafter, to assure the continuity of the stabilizer u, we will replace $\xi_\alpha(x)$ by $\xi_\alpha^{\epsilon}(x)$ for any $\epsilon > 0$. Accordingly, we will replace $a_1(x), a_2(x)$, Inequalities (17) and (18) respectively by

$$\begin{aligned} &a_1^{\epsilon}(x) = a(x) - \eta(x) + \mu(x)\xi_\alpha^{\epsilon}(x), \\ &a_2^{\epsilon}(x) = \eta(x) - \mu(x)\xi_\alpha^{\epsilon}(x), \\ &\inf_{u_I \in \mathrm{Im}(R(x))} [a_1^{\epsilon}(x) + b^{\mathrm{T}}(x)u_I + u_I^{\mathrm{T}} R(x) u_I] < 0, \\ &\forall x \neq 0, \end{aligned} \tag{27}$$

and

$$\inf_{u_P \in \mathrm{Im}^{\perp}(R(x))} [a_2^{\epsilon}(x) + b^{\mathrm{T}}(x) u_P] < 0, \quad \forall x \neq 0. \tag{28}$$

3. Under the above replacements, the feasible sets Φ_μ^α and Ψ_μ^α will be replaced respectively by

$$\begin{aligned} \Phi_\mu^{\epsilon,\alpha} = \Big\{ &u_I(x) \in \mathrm{Im}(R(x)) \Big| a_1^{\epsilon}(x) + b^{\mathrm{T}}(x) u_I(x) \\ &+ u_I^{\mathrm{T}}(x) R(x) u_I(x) < 0, \forall x \neq 0; u_I(0) = 0 \Big\}, \end{aligned}$$

and

$$\begin{aligned} \Psi_\mu^{\epsilon,\alpha} = \Big\{ &u_P(x) \in \mathrm{Im}^{\perp}(R(x)) \Big| a_2^{\epsilon}(x) + b^{\mathrm{T}}(x) u_P < 0, \\ &\forall x \neq 0; u_P(0) = 0 \Big\}. \end{aligned}$$

Moreover, Equality (20) in Theorem 12 becomes

$$\begin{aligned} \Phi = \bigcup_{\epsilon > 0} \bigcup_{\alpha < 0} \bigcup_{\mu(x) \in \mathcal{D}} \{ &u = u_I(x) + u_P(x) \,| \\ &u_I(x) \in \Phi_\mu^{\epsilon,\alpha}, u_P(x) \in \Psi_\mu^{\epsilon,\alpha} \}. \end{aligned} \tag{29}$$

Set $v = R^{\frac{1}{2}}(x) u_I$. Note that $\mathrm{Im}(R^{\frac{1}{2}}(x)) = \mathrm{Im}(R(x))$, then by Lemma 14, $u_I = (R^{\frac{1}{2}})^{+}(x) v \in \mathrm{Im}(R(x))$. Moreover, for each $u_I \in \mathrm{Im}(R(x))$, we can find v such that $u_I = (R^{\frac{1}{2}})^{+}(x) v$. So we can convert (27) to the following

$$\inf_{v \in \mathbb{R}^m} [a_1^{\epsilon}(x) + \tilde{b}^{\mathrm{T}}(x) v + v^{\mathrm{T}} v] < 0, \quad \forall x \neq 0, \tag{30}$$

where $\tilde{b}(x) = (R^{\frac{1}{2}})^{+}(x) b(x)$. That is, the solutions of (30) and the solutions of (27) are one-to-one corresponded by $v \leftrightarrow u_I = (R^{\frac{1}{2}})^{+}(x) v$.

Now getting the parameterized formula of $v(x)$ satisfying (30) is similar to the case of $R(x) > 0, \forall x \neq 0$, which was discussed in the previous section. We have

Theorem 19. *Assume V is a CLF for system (1), and Assumptions 8 and 15 hold. Then the set of solutions $u_I = u_I(x)$ is given by*

$$\begin{aligned}u_I(x) &= -\frac{1}{2}R^{+}(x)b(x)\\ &\quad + (R^{\frac{1}{2}})^{+}(x)\nu(x)\sqrt{\frac{1}{4}b^{\mathrm{T}}(x)R^{+}(x)b(x) - a_1^{\epsilon}(x)},\\ &\quad \nu(x)\in\mathcal{B},\quad \forall x\neq 0,\end{aligned} \tag{31}$$

and the set of continuous (over $\mathbb{R}^n \setminus \{0\}$) solutions $u_I(x)$ is parameterized by (31) with any continuous (over $\mathbb{R}^n \setminus \{0\}$) function $\nu(x) \in \mathcal{B}$. Moreover, if the CLF V for system (1) satisfies the small control property, then $u_I(x)$ is continuous over $\mathbb{R}^n$ with $u_I(0) = 0$.

Proof. Since V is a CLF for system (1), based on the above argument, (30) holds. In turn, there exits a $v \in \mathbb{R}^m$ such that

$$a_1^{\epsilon}(x) + \tilde{b}^{\mathrm{T}}(x)v + v^{\mathrm{T}}v < 0, \quad \forall x \neq 0.$$

Using Lemma 6, we have

$$\begin{aligned}v &= -\frac{1}{2}\tilde{b}(x) + \nu(x)\sqrt{\frac{1}{4}\tilde{b}^{\mathrm{T}}(x)\tilde{b}(x) - a_1^{\epsilon}(x)}\\ &= -\frac{1}{2}(R^{\frac{1}{2}})^{+}(x)b(x) + \nu(x)\sqrt{\frac{1}{4}b^{\mathrm{T}}(x)R^{+}(x)b(x) - a_1^{\epsilon}(x)},\\ &\quad \nu(x)\in\mathcal{B},\quad \forall x\neq 0.\end{aligned}$$

Then $u_I = (R^{\frac{1}{2}})^{+}(x)v$ yields (31). □

Next, we derive u_P from (28). We will also relax the restriction on u_P.

It follows that (28) is equivalent to

$$\inf_{v\in\mathbb{R}^m} [a_2^{\epsilon}(x) + \bar{b}^{\mathrm{T}}(x)v] < 0, \tag{32}$$

where $\bar{b}(x) = (I - R^{+}(x)R(x))b(x)$.

The parameterized formula for all the solutions can be obtained by the method proposed in Curtis and Beard (2004). Following the notations and terminologies, we briefly sketch the construction of the set of controls using the techniques proposed in Curtis and Beard (2004).

Choose selectability: $p_s(v, x) = -[a_2^{\epsilon}(x)+\bar{b}^{\mathrm{T}}(x)v]$, and the rejectability: $p_r(v, x) = l(x) + v^{\mathrm{T}}Q(x)v$, where the function $l(x) \geq 0$ with $l(0) = 0$ and the matrix function $Q(x) > 0$, and selectivity index: the function $\gamma(x)$ satisfying $0 < \gamma(x) < \infty$. Then the feasible set is expressed as

$$\mathcal{S}_{\gamma(x)}(x) = \left\{ v\in\mathbb{R}^m \,\middle|\, -[a_2^{\epsilon}(x) + \bar{b}^{\mathrm{T}}(x)v] > \frac{1}{\gamma(x)}(l(x) + v^{\mathrm{T}}Q(x)v), \forall x\neq 0 \right\}. \tag{33}$$

By Lemma 6, it follows that

$$\mathcal{S}_{\gamma(x)}(x) = \left\{ -\frac{1}{2}\gamma(x)Q^{-1}(x)\bar{b}(x) + Q^{-\frac{1}{2}}(x)\nu(x) \cdot \sqrt{\frac{1}{4}\gamma^2(x)\bar{b}^{\mathrm{T}}(x)Q^{-1}(x)\bar{b}(x) - l(x) - \gamma(x)a_2^{\epsilon}(x)} \right\}.$$

Define

$$\underline{\gamma}(x) = \begin{cases} -\dfrac{1}{a_2^{\epsilon}(x)}, & \text{if } \bar{b}^{\mathrm{T}}(x) = 0,\\[2ex] \dfrac{2a_2^{\epsilon}(x) + 2\sqrt{(a_2^{\epsilon}(x))^2 + l(x)\bar{b}^{\mathrm{T}}(x)Q^{-1}(x)\bar{b}(x)}}{\bar{b}^{\mathrm{T}}(x)Q^{-1}(x)\bar{b}(x)}, & \text{otherwise.}\end{cases} \tag{34}$$

According to Lemma 6 in Curtis and Beard (2004), $\gamma(x) > \underline{\gamma}(x)$ implies $\mathcal{S}_{\gamma(x)}$ is nonempty.

Let

$$\sigma_1(x, \gamma(x)) = \frac{1}{2}\gamma(x)Q^{-1}(x)\bar{b}(x), \tag{35}$$

and

$$\sigma_2(x, \gamma(x)) = Q^{-\frac{1}{2}}(x) \cdot \sqrt{\frac{1}{4}\gamma^2(x)\bar{b}^{\mathrm{T}}(x)Q^{-1}(x)\bar{b}(x) - l(x) - \gamma(x)a_2^{\epsilon}(x)}, \tag{36}$$

then the feasible set becomes

$$\begin{aligned}\mathcal{S}(x) &= \bigcup_{\gamma(x)>\underline{\gamma}(x)} \mathcal{S}_{\gamma(x)}(x)\\ &= \Big\{ -\sigma_1(x,\gamma(x)) + \sigma_2(x,\gamma(x))\nu(x) \,|\, \gamma(x) > \underline{\gamma}(x), \|\nu(x)\| < 1, \forall x\neq 0 \Big\}.\end{aligned} \tag{37}$$

Moreover, if the CLF V satisfies the small control property, and $\gamma(x) = \zeta(x)\underline{\gamma}(x)$ with $1 < \zeta(x) < N < +\infty$, and $Q(x)$ satisfies $\underline{r}I \leq Q(x) \leq \bar{r}I, \forall x \in \mathbb{R}^n$, where $\underline{r}$ and $\bar{r}$ are positive constants, then each $k(x) \in \mathcal{S}(x)$ is also continuous at origin with $k(0) = 0$.

Summarizing the above arguments, we have

Theorem 20. *Assume V is a CLF for system (1), and Assumptions 8 and 15 hold, then a set of satisficing controls $u_P = u_P(x)$ is given by:*

$$\begin{aligned}u_P(x) &= -(I - R^{+}(x)R(x))\sigma_1(x,\gamma(x))\\ &\quad + (I - R^{+}(x)R(x))\sigma_2(x,\gamma(x))\nu(x),\\ &\quad \gamma(x) \geq \underline{\gamma}(x), \|\nu(x)\| < 1, \quad \forall x\neq 0,\end{aligned} \tag{38}$$

which is continuous over $\mathbb{R}^n \setminus \{0\}$ with any continuous (over $\mathbb{R}^n \setminus \{0\}$) function $\nu(x) \in \mathcal{B}$. Moreover, if the CLF V satisfies the small control property, and $\gamma(x) = \zeta(x)\underline{\gamma}(x)$ with $1 < \zeta(x) < N < +\infty$, and $Q(x)$ satisfies $\underline{r}I \leq Q(x) \leq \bar{r}I, \forall x \in \mathbb{R}^n$, where $\underline{r}$ and $\bar{r}$ are positive constants, then $u_P(x)$ is continuous at the origin, thus over $\mathbb{R}^n$ with $u_P(0) = 0$.

Remark 21. If $\underline{\gamma}(x)$ is continuous, then the construction of $\gamma(x)$ will be much easier. Lemma 6 in Curtis and Beard (2004) proved that as long as $l(x)$ satisfies the property

$$(\bar{b}^{\mathrm{T}}(x) \neq 0 \quad \text{and} \quad a_2^{\epsilon}(x) = 0) \Rightarrow l(x) > 0,$$

then $\underline{\gamma}(x)$ is continuous over $\mathbb{R}^n \setminus \{0\}$. So we may simply choose $l(x) = \bar{b}^{\mathrm{T}}(x)\bar{b}(x)$, which ensures the continuity of $\underline{\gamma}(x)$.

5. Remarks on assumptions

In this section we investigate some assumptions used in our previous discussion and relax some restrictions.

First, we consider the case when $R^+(x)$ (or equivalently, $(R^{\frac{1}{2}})^+(x)$) is continuous. We define

Definition 22. Let $R(x)$ be an $n \times n$ symmetric matrix with continuous entries. A point x is a regular point of $R(x)$ if there exists a neighborhood U of x, such that $\mathrm{rank}(R(x)) = $ constant, $\forall x \in U$. Otherwise, x is called a singular point. $R(x)$ is said to be regular, if each $x \neq 0$ is a regular point for $R(x)$.

Lemma 23. *$R^+(x)$ is continuous at x_0, if and only if x_0 is a regular point.*

Proof. Denote by $\mathcal{N}_{x_0}$ the set of neighborhoods of x_0, then we can define the neighbor rank of x_0 as

$$r_{\mathcal{N}}(x_0) = \max_{x \in \mathcal{N}} \mathrm{rank}(R(x))$$

with $\mathcal{N} \in \mathcal{N}_{x_0}$. Assume $r_{\mathcal{N}}(x_0) = s$, then we can express $R(x)$ around x_0 (over some neighborhood $\mathcal{N}$) as

$$P(x)R(x)P^{\mathrm{T}}(x) = \mathrm{diag}(l_1(x), \ldots, l_s(x), 0, \ldots, 0)$$

with some orthogonal matrix $P(x)$. Since

$$P(x)R^+(x)P^{\mathrm{T}}(x) = \mathrm{diag}\left(\frac{1}{l_1(x)}, \ldots, \frac{1}{l_s(x)}, 0, \ldots, 0\right),$$

it is easy to see that $R^+(x)$ is not continuous, if and only if $l_i(x_0) = 0$ for some $1 \le i \le s$. $\square$

Next, we consider Assumption 8, i.e., $R(x) \ge 0$, $\forall x \neq 0$. Lemma 4 already answered this question partly. To make it clearer, let $R(x) = R_p(x) - R_n(x)$, where $R_p(x)$ is the positive part and $-R_n(x)$ is the negative part for all $x \in \mathbb{R}^n \setminus \{0\}$. More precisely, let

$$P(x)R(x)P^{\mathrm{T}}(x) = \mathrm{diag}(l_1(x), \ldots, l_t(x), l_{t+1}(x), \ldots, l_{t+s}(x), 0, \ldots, 0),$$

where $P(x)$ is an orthogonal matrix, $l_i(x) > 0$, $i = 1, \ldots, t$, and $l_i(x) < 0$, $i = t+1, \ldots, t+s$, $\forall x \neq 0$, then

$$R_p(x) = P^{\mathrm{T}}(x)\mathrm{diag}(l_1(x), \ldots, l_t(x), 0, \ldots, 0)P(x),$$

and

$$R_n(x) = -P^{\mathrm{T}}(x)\mathrm{diag}(0, \ldots, 0, l_{t+1}(x), \ldots, l_{t+s}(x), 0, \ldots, 0)P(x).$$

Assume $R_n(x) \neq 0$. We split u as $u = u_p + u_n + u_z$, where

$$u_p \in \mathrm{Im}(R_p(x)),\ u_n \in \mathrm{Im}(R_n(x)),\ u_z \in \mathrm{Im}^{\perp}(R(x)).$$

Then

$$\begin{aligned} &a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}}R(x)u \\ &= a(x) + b^{\mathrm{T}}(x)(u_p + u_n + u_z) + u_p^{\mathrm{T}}R_p(x)u_p - u_n^{\mathrm{T}}R_n(x)u_n. \end{aligned}$$

Simply set $u_p = 0$, $u_z = 0$ and define $v = R_n^{\frac{1}{2}}(x)u_n$. Then we have $u_n = (R_n^{\frac{1}{2}})^+(x)v$, and

$$\begin{aligned} &a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}}R(x)u \\ &\quad = a(x) + b^{\mathrm{T}}(x)(R_n^{\frac{1}{2}})^+(x)v - v^{\mathrm{T}}v \\ &\quad := a(x) + \tilde{b}^{\mathrm{T}}(x)v - v^{\mathrm{T}}v < 0, \quad \forall x \neq 0. \end{aligned}$$

Let $\nu(x), p(x) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m$ be two continuous vectors satisfying $\frac{1}{2}\tilde{b}(x) \neq p(x) \in \mathrm{Im}(R_n(x))$, $\nu(x) \in (\bar{\mathcal{B}})^c$, i.e., $\|\nu(x)\| > 1$. Then the set of solutions of v can be expressed as

$$v = \begin{cases} p(x), \quad \text{if } \frac{1}{4}\tilde{b}^{\mathrm{T}}(x)\tilde{b}(x) + a(x) \le 0, \\ \frac{1}{2}\tilde{b}(x) + \nu(x)\sqrt{\frac{1}{4}\tilde{b}^{\mathrm{T}}(x)\tilde{b}(x) + a(x)}, \\ \quad \text{if } \frac{1}{4}\tilde{b}^{\mathrm{T}}(x)\tilde{b}(x) + a(x) > 0. \end{cases} \tag{39}$$

By Lemma 23, the above argument leads to:

Theorem 24. *Assume V is a CLF for system (1), and $R_n(x) \neq 0$ is regular. Then u_n can be constructed to stabilize system (1). The general expression of the set of stabilizing controls is*

$$u_n = \begin{cases} (R_n^{\frac{1}{2}})^+(x)p(x), \\ \quad \text{if } \frac{1}{4}b^{\mathrm{T}}(x)R_n^+(x)b(x) + a(x) \le 0, \\ \frac{1}{2}R_n^+(x)b(x) \\ \quad + (R_n^{\frac{1}{2}})^+(x)\nu(x)\sqrt{\frac{1}{4}b^{\mathrm{T}}R_n^+(x)b(x) + a(x)}, \\ \quad \text{if } \frac{1}{4}b^{\mathrm{T}}(x)R_n^+(x)b(x) + a(x) > 0, \end{cases} \tag{40}$$

where $\nu(x) \in (\bar{\mathcal{B}})^c$, $\frac{1}{2}(R_n^{\frac{1}{2}})^+(x)b(x) \neq p(x) \in \mathrm{Im}(R_n(x))$.

Finally, we give an example.

Example 25. Consider the following system

$$\begin{cases} \dot{x}_1 = x_1^5 + 3x_1^3u_1 + x_2^2u_2 + x_1u_1^2 \\ \dot{x}_2 = 2x_2^3u_1 + x_1x_2u_2 + x_2u_1^2 \end{cases} \tag{41}$$

where $x = [x_1\ x_2]^{\mathrm{T}}$ is the state, and $u = [u_1\ u_2]^{\mathrm{T}}$ is the control input.

Take $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. Then

$$\dot{V}|_{(41)} = x_1^6 + (3x_1^4 + 2x_2^4)u_1 + 2x_1x_2^2u_2 + (x_1^2 + x_2^2)u_1^2.$$

So

$$a(x) = x_1^6, \qquad b(x) = \begin{bmatrix} 3x_1^4 + 2x_2^4 \\ 2x_1x_2^2 \end{bmatrix},$$

$$R(x) = \begin{bmatrix} x_1^2 + x_2^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to see that $R(x) \ge 0$, $\forall x \in \mathbb{R}^2$. Setting

$$u = \begin{cases} \begin{bmatrix} -\dfrac{3x_1^4 + 2x_2^4}{2(x_1^2 + x_2^2)} & 0 \end{bmatrix}^{\mathrm{T}}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

we can prove that V is a CLF for system (41) and satisfies the small control property. Clearly,

$$R^+(x) = \mathrm{diag}(1/(x_1^2+x_2^2), 0), \quad \forall x \neq 0,$$

and

$$(R^{\frac{1}{2}})^+(x) = \mathrm{diag}(1/\sqrt{x_1^2+x_2^2}, 0), \quad \forall x \neq 0,$$

$R^+(x)$ and $(R^{\frac{1}{2}})^+(x)$ are both continuous for any $x \in \mathbb{R}^2 \setminus \{0\}$. Moreover,

$$\mathrm{Im}(R(x)) = \mathrm{Span}\left\{[1\ 0]^{\mathrm{T}}\right\},$$
$$\mathrm{Im}^{\perp}(R(x)) = \mathrm{Span}\left\{[0\ 1]^{\mathrm{T}}\right\},$$
$$b_I(x) = [3x_1^4+2x_2^4\ \ 0]^{\mathrm{T}}, \qquad b_P(x) = [0\ \ 2x_1x_2^2]^{\mathrm{T}},$$

and

$$\mathcal{S} = \{x \in \mathbb{R}^2 \setminus \{0\} | x_1 = 0 \text{ or } x_2 = 0\}.$$

Straightforward calculations lead to

$$\eta(x) = \begin{cases} -\dfrac{5x_1^8+12x_1^4x_2^4+4x_2^8-4x_1^6x_2^2}{4(x_1^2+x_2^2)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and

$$\xi(x) = \begin{cases} -\infty, & \text{if } x_1 \neq 0 \text{ and } x_2 \neq 0, \\ \eta(x), & \text{if } x_1 = 0 \text{ or } x_2 = 0. \end{cases}$$

Using the notations in Section 4, we have

$$a_1^{\epsilon}(x) = \frac{(3x_1^4+2x_2^4)^2}{4(x_1^2+x_2^2)} + \mu(x)\xi_{\alpha}^{\epsilon}(x), \quad \forall x \neq 0,$$

and

$$a_2^{\epsilon}(x) = -\frac{5x_1^8+12x_1^4x_2^4+4x_2^8-4x_1^6x_2^2}{4(x_1^2+x_2^2)} - \mu(x)\xi_{\alpha}^{\epsilon}(x), \quad \forall x \neq 0, \tag{42}$$

and

$$\bar{b}(x) = [0\ \ 2x_1x_2^2]^{\mathrm{T}}, \quad \forall x \neq 0. \tag{43}$$

According to Theorem 19, we obtain the expression of $u_I = u_I(x) \in \mathrm{Im}(R(x))$ as

$$u_I(x) = \begin{bmatrix} -\dfrac{3x_1^4+2x_2^4}{2(x_1^2+x_2^2)} + \nu_1(x)\sqrt{\dfrac{(3x_1^4+2x_2^4)^2}{4(x_1^2+x_2^2)} - a_1^{\epsilon}(x)} \\ 0 \end{bmatrix}, \quad \forall x \neq 0,$$

where the continuous function $\nu_1(x)$ satisfies $|\nu_1(x)| < 1, \forall x \in \mathbb{R}^2 \setminus \{0\}$.

On the other hand, according to Theorem 20 and with its notations, we obtain $u_P = u_P(x) \in \mathrm{Im}^{\perp}(R(x))$ as

$$u_P(x) = -\frac{1}{2}\gamma(x)(I - R^+(x)R(x))Q^{-1}(x)\bar{b}(x) + (I - R^+(x)R(x))Q^{-\frac{1}{2}}(x)\nu(x) \cdot \sqrt{\frac{1}{4}\gamma^2(x)\bar{b}^{\mathrm{T}}(x)Q^{-1}(x)\bar{b}(x) - l(x) - \gamma(x)a_2^{\epsilon}(x)}, \quad \forall x \neq 0,$$

where $a_2^{\epsilon}(x)$ and $\bar{b}(x)$ are ones in (42) and (43) respectively, and

$$I - R^+(x)R(x) = \mathrm{diag}(0, 1), \quad \forall x \neq 0,$$

and $\gamma(x) \geq \underline{\gamma}(x), \forall x \neq 0$, with

$$\underline{\gamma}(x) = \begin{cases} -\dfrac{1}{a_2^{\epsilon}(x)}, & \text{if } x_1 = 0 \text{ or } x_2 = 0, \\ \dfrac{2a_2^{\epsilon}(x) + 2\sqrt{(a_2^{\epsilon}(x))^2 + l(x)\bar{b}^{\mathrm{T}}(x)Q^{-1}(x)\bar{b}(x)}}{\bar{b}^{\mathrm{T}}(x)Q^{-1}(x)\bar{b}(x)}, \\ \quad \text{if } x_1 \neq 0 \text{ and } x_2 \neq 0. \end{cases}$$

For example, we can choose $Q(x) \equiv I$ and $l(x) \equiv 0$, then the above $u_P = u_P(x)$ is reduced to the following

$$u_P(x) = \begin{bmatrix} 0 \\ -\gamma(x)x_1x_2^2 + \nu_2(x)\sqrt{\gamma^2(x)x_1^2x_2^4 - \gamma(x)a_2^{\epsilon}(x)} \end{bmatrix}, \quad \forall x \neq 0$$

with the continuous function $\nu_2(x)$ satisfies $|\nu_2(x)| < 1, \forall x \neq 0$, and $\gamma(x) \geq \underline{\gamma}(x), \forall x \neq 0$, with

$$\underline{\gamma}(x) = \begin{cases} -\dfrac{1}{a_2^{\epsilon}(x)}, & \text{if } x_1 = 0 \text{ or } x_2 = 0, \\ \dfrac{2a_2^{\epsilon}(x) + 2\left|a_2^{\epsilon}(x)\right|}{4x_1^2x_2^4}, & \text{if } x_1 \neq 0 \text{ and } x_2 \neq 0. \end{cases}$$

Therefore, system (41) is globally asymptotically stabilizable by the state feedback control law $u = u_I(x) + u_P(x)$ with $u(0) = 0$ which is continuous except possibly at $x = 0$.

6. Design of piecewise continuous controls

From the above discussion we see that under the assumption of the regularities of $R_n(x)$ and $R_p(x)$, we are able to construct continuous controls. Unfortunately, it is, in general, a quite restrictive assumption. In this section, we give a method to extend the results of previous sections to piecewise continuous controls.

In fact, from the Lyapunov approach point of view, we have two ways to find a Lyapunov function: One way is to find a control first and then find a Lyapunov function for the closed-loop system. The other one is to find a CLF and then construct a control. It is obvious that finding a CLF has much more freedom than finding a Lyapunov function. So we may consider a stabilizing problem by choosing a candidate CLF and then checking whether it satisfies the requirement of the CLF for the system by constructing a stabilizing control.

Based on this consideration, we propose the following way to construct piecewise continuous controls. For statement ease, we first denote $R(x) = R_p(x) - R_n(x)$ with the non-zero eigenvalues of $R_p(x)$ and $R_n(x)$ being positive and negative respectively.

- Step 1: Construct a stabilizing control on
  $$\mathcal{D}_1 = \{x \in \mathbb{R}^n \setminus \{0\} \big| R_n(x) \neq 0\},$$
  by using formula (40).
- Step 2: Construct a stabilizing control on
  $$\begin{aligned}\mathcal{D}_2 &= \{x \in \mathbb{R}^n \setminus \{0\} \big| R_n(x) = 0\} \\ &= \{x \in \mathbb{R}^n \setminus \{0\} \big| R(x) \geq 0\},\end{aligned}$$
  by using the method developed in Sections 3 and 4.

In each step, we consider the regular region first, and then the singular region. Note that because of the natural robustness provided by the CLF, a stabilizing control working on the singular region works over a neighborhood of any point in the region, thus it can cover the "bad area" of the regular region. In fact, in many cases we can piece the stabilizing controls together to produce a continuous control. But in this section, we do not pursue this.

We give a simple example to illustrate this.

Example 26. Consider the following system

$$\begin{cases}\dot{x}_1 = x_2 + x_1^2 \\ \dot{x}_2 = x_3^2 + u_1 \\ \dot{x}_3 = x_3 x_1^2 - x_3 u_2^2 + x_1 u_2.\end{cases} \tag{44}$$

Choose a candidate CLF as:

$$V(x) = 5x_1^2 + 2x_1(x_2 + x_1^2) + (x_2 + x_1^2)^2 + x_3^2.$$

We do not need any pre-knowledge that it is a CLF. By constructing a stabilizing control piece by piece, we can finally verify whether it is a CLF.

Calculate that

$$\dot{V}|_{(44)} = a(x) + b^{\mathrm{T}}(x)u + u^{\mathrm{T}} R(x) u,$$

where

$$\begin{aligned} a(x) &= (10x_1 + 4x_1^2 + 2x_3^2)(x_2 + x_1^2) \\ &\quad + (4x_1 + 2)(x_2 + x_1^2)^2 + 2x_1 x_3^2 + 2x_3^2 x_1^2, \end{aligned}$$

$$b(x) = [2(x_1 + x_2 + x_1^2) \;\; 2x_1 x_3]^{\mathrm{T}}, \quad \text{and}$$

$$R(x) = \mathrm{diag}(0, \; -2x_3^2).$$

It is clear that

$$\mathcal{D}_1 = \{x \in \mathbb{R}^3 \setminus \{0\} | x_3 \neq 0\}.$$

Over $\mathcal{D}_1$ we can use (40) to construct control u_n. By choosing any $\beta(x) > 0$, $\beta(x) \geq \frac{|x_1|+1}{\sqrt{2}}$ and setting $p(x) = [0 \; \beta(x)]^{\mathrm{T}}$, $\nu(x) = [1 + \beta(x) \;\; 1 + \beta(x)]^{\mathrm{T}}$, we can construct control $u_n = [0 \; u_2]^{\mathrm{T}}$ over $\mathcal{D}_1$ as

$$u_2 = \begin{cases} \dfrac{\beta(x)}{\sqrt{2}|x_3|}, & \text{if } \dfrac{1}{2}x_1^2 + a(x) \leq 0, \\ \dfrac{x_1}{2x_3} + \dfrac{1 + \beta(x)}{\sqrt{2}|x_3|}\sqrt{\dfrac{1}{2}x_1^2 + a(x)}, & \text{if } \dfrac{1}{2}x_1^2 + a(x) > 0. \end{cases} \tag{45}$$

Since $\mathcal{D}_2 = \{x \in \mathbb{R}^3 \setminus \{0\} | x_3 = 0\}$, on $\mathcal{D}_2$ we have

$$\begin{aligned} a(x) &= (10x_1 + 4x_1^2)(x_2 + x_1^2) \\ &\quad + (4x_1 + 2)(x_2 + x_1^2)^2, \end{aligned} \tag{46}$$

$$b(x) = [2(x_1 + x_2 + x_1^2) \;\; 0]^{\mathrm{T}}, \quad \text{and} \quad R_p(x) = 0.$$

Now since $R(x) = R_p(x) = 0$, it is degenerated to the affine case. We simply need to check whether $a(x) < 0$, for $x \neq 0$ as $b(x) = 0$. The answer is yes, because $b(x) = 0$ implies

$$x_2 = -x_1 - x_1^2, \tag{47}$$

and then

$$a(x) = -8x_1^2 \leq 0. \tag{48}$$

However, $a(x) = 0$ implies $x_1 = 0$, and thus by (47) $x_2 = 0$, which leads to the conclusion that $V(x)$ is a control Lyapunov function.

Now constructing u_1 becomes standard. Using Sontag's formulas in Sontag (1989), we have

$$u_1(x) = \begin{cases} \dfrac{a(x) + \sqrt{a^2(x) + 4(x_1 + x_2 + x_1^2)^2}}{2(x_1 + x_2 + x_1^2)}, \\ \quad \text{if } x_1 + x_2 + x_1^2 \neq 0, \\ 0, \quad \text{otherwise}, \end{cases} \tag{49}$$

where $a(x)$ is the one in (46). We do not care the value at $x = 0$.

We conclude that the stabilizing control is: on $\mathcal{D}_1$, $u_1 = 0$ and u_2 as in (45), and on $\mathcal{D}_2$, u_1 as in (49) and $u_2 = 0$. □

In fact, the control on singular points ($\mathcal{D}_2$) is always robust, because of the property of the CLF. So we can choose a small enough $\epsilon > 0$, such that on the ϵ-neighborhood of $\mathcal{D}_2$ we use the control defined on $\mathcal{D}_2$. (ϵ could be state-depending.) By this way we can avoid the "blow up" of the control. That is, in the Example 26, we use control u_2 over $\{x \in \mathbb{R}^3 \setminus \{0\} \big| |x_3| > \epsilon\}$, and use u_1 over $\{x \in \mathbb{R}^3 \setminus \{0\} \big| |x_3| \leq \epsilon\}$.

7. Concluding remarks

In this paper we considered the problem of constructing stabilizing controls for quadratic input nonlinear systems via a known control Lyapunov function. Under the regularity assumption we provided a constructive parameterization of the class of continuous universal controls that render the system globally asymptotically stable. For the more general case, certain set of piecewise continuous stabilizing controls was constructed. Even without a given CLF, the method could, through constructing stabilizing controls, verify whether a candidate CLF is a real CLF.

The following two problems are interesting for further investigation:

1. If the method can be used for stabilization of switched nonlinear systems. Consider a switched nonlinear system with finite switching modes:
   $$\dot{x} = f_{\sigma(x)}(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \tag{50}$$
   where $\sigma(x) : \mathbb{R}^n \to Ł = \{1, 2, \ldots, N\}$. If the switching law is controllable, we may define a control Lyapunov function

V as in Sun and Zhao (2001): a smooth, proper, positive definite function V satisfying

$$\min_{l \in \text{Ł}} \inf_{u \in \mathbb{R}^m} \frac{\partial V}{\partial x}[f_l(x,u)] < 0, \quad \forall x \neq 0.$$

If such a V exists, we can first use the adaptive way to construct switching law $\sigma(x) = l^*$, where

$$\inf_{u \in \mathbb{R}^m} \frac{\partial V}{\partial x}[f_{l^*}(x,u)] = \min_{l \in \text{Ł}} \inf_{u \in \mathbb{R}^m} \frac{\partial V}{\partial x}[f_l(x,u)], \quad \forall x \neq 0. \tag{51}$$

Then for the fixed l^* the control design technique proposed in this paper can be used to design the stabilizing control. As soon as the stabilizing control fails to work, we can use (51) to switch the mode. Some preliminary simulations show that this method is promising.

2. For quadratic input systems, which imply certain convexity, piecewise continuous controls might be pieced together to form a continuous stabilizing control. There are some examples in which it is not difficult to find the "gluing function". However, finding a general rule is rather difficult.

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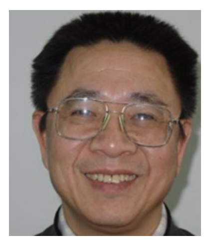

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