Available online at www.sciencedirect.com
SCIENCE DIRECT®

Automatica 40 (2004) 439 – 447

automatica

www.elsevier.com/locate/automatica

Brief paper

Non-regular feedback linearization of nonlinear systems via a normal form algorithm☆

Daizhan Cheng[a,*], Xiaoming Hu[b], Yuzhen Wang[c]

[a] Institute of Systems Sciences, Chinese Academy of Sciences, Beijing 100080, PR China
[b] Optimization and Systems Theory, Royal Institute of Technology, Stockholm, Sweden
[c] School of Control Science & Engineering, Shandong University, Jinan 250061, PR China

Received 15 April 2003; received in revised form 22 October 2003; accepted 26 October 2003

Abstract

In this paper, the problem of non-regular static state feedback linearization of affine nonlinear systems is considered. First of all, a new canonical form for non-regular feedback linear systems is proposed. Using this form, a recursive algorithm is presented, which yields a condition for single input linearization. Then the left semi-tensor product of matrices is introduced and several new properties are developed. Using the recursive framework and new matrix product, a formula is presented for normal form algorithm. Based on it, a set of conditions for single-input (approximate) linearizability is presented.

Keywords: Linearization; Singular state feedback; Normal form; Semi-tensor product of matrices

1. Introduction

Consider an affine nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad f(0) = 0, \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m. \quad (1)$$

Throughout the paper the vector fields $f(x)$, $g_i(x)$ and controls, etc. are assumed to be analytic (C^ω) to assure the convergence of the Taylor series expansion of the vector fields, etc. A general linearization problem is defined as follows (Ge, Sun, & Lee, 2001; Guay, 2002).

Definition 1.1. System (1) is non-regular state feedback linearizable at the origin, if there exist a feedback control

$$u = \alpha(x) + \beta(x)v, \quad (2)$$

with $m \times k$ matrix $\beta(x)$ and a diffeomorphism $z = \xi(x)$ such that in coordinate frame z the closed-loop system can be expressed as a completely controllable linear system. If $\beta(x)$ is a square non-singular matrix, it is called the regular state feedback linearization. When $k = 1$ the linearization is called the single-input linearization.

Non-regular state feedback linearization has recently been investigated in Sun and Xia (1997), Ge et al. (2001). The following lemma, which simplified the study of non-regular state feedback linearization, is an immediate consequence of Heymann's Lemma (Heymann, 1968; Desoer & Vidyasagar, 1975).

Lemma 1.2. *System* (1) *is non-regular state feedback linearizable, iff it is single-input linearizable, i.e., linearizable by control* (2) *with* $m \times 1$ *vector* $\beta(x)$.

The following lemma is useful in the sequel.

Lemma 1.3 (Sun & Xia, 1997). *Let* $A = J_f(0)$ *be the Jacobian matrix of* f *at the origin,* $B = g(0)$. *If system* (1) *is linearizable, then* (A, B) *is completely controllable.*

Normal forms have been used to investigate various control problems of nonlinear systems (Cheng & Martin, 2002;

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Jessy W. Grizzle under the direction of Editor Hassan Khalil. Supported partly by CNSF G60274010, G60343001, National 973 Project of China G1998020308 and partly by Sida-VR Swedish Research Links Grant 348-2002-6936.
* Corresponding author. Tel.: +86-10-6265-1445; fax: +86-10-6258-7343.
E-mail addresses: dcheng@iss03.iss.ac.cn (D. Cheng), hu@math.kth.se (X. Hu), yzwang@tsinghua.edu.cn (Y. Wang).

0005-1098/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2003.10.014

Devanathan, 2001; Krener & Kang, 1990). We first review some concepts (Guckenheimer & Holmes, 1983): Let H_n^k be the set of kth degree homogeneous polynomial vector fields in R^n. Then the following facts are obvious:

(1) H_n^k is a linear vector space over $\mathbb{R}$.
(2) Let $Ax \in H_n^1$ be a given vector field with A an $n \times n$ constant matrix. Then the Lie derivative $ad_{Ax} : H_n^k \to H_n^k$ is a linear mapping.

The following normal form representation (Arnold, 1983) and its application to linearization (Devanathan, 2001) are the starting point of our approach.

Theorem 1.4 (Poincare's Theorem (Arnold, 1983)). *Consider a C^ω dynamic system*

$$\dot{x} = Ax + f_2(x) + f_3(x) + \cdots, \quad x \in \mathbb{R}^n, \tag{3}$$

where $f_i(x)$, $i \geqslant 2$ are ith degree homogeneous vector fields. If A is non-resonant, there exists a formal change of coordinates

$$x = y + h(y), \tag{4}$$

where $h(y)$ corresponds to the sum of possibly infinite homogeneous vector polynomials $h_m(y)$, $m \geqslant 2$, that is $h(y) = h_2(y) + h_3(y) + \cdots$, such that system (3) *can be expressed as $\dot{y} = Ay$.*

This has been extended to single input control systems (Krener, Karahan, & Hubbard, 1988).

We recall that for a matrix A, let $\sigma(A) = \lambda = (\lambda_1, \ldots, \lambda_n)$ be its eigenvalues. A is a *resonant* matrix if there exists $m = (m_1, \ldots, m_n) \in Z_+^n$, and $|m| \geqslant 2$, i.e., $m_i \geqslant 0$ and $\sum_{i=1}^n m_i \geqslant 2$, such that for some s, $\lambda_s = \langle m, \lambda \rangle$. The following proposition provides a sufficient condition for non-resonance.

Proposition 1.5 (Devanathan, 2001). *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of a given Hurwitz matrix A. A is non-resonant if*

$$\max\{|Re(\lambda_i)| \mid \lambda_i \in \sigma(A)\} \leqslant 2\min\{|Re(\lambda_i)| \mid \lambda_i \in \sigma(A)\}. \tag{5}$$

In this paper we use the normal form theory to investigate the non-regular state feedback (approximate) linearization. First, we propose a new feedback canonical form and a recursive result for single-input linearization. Then the left semi-tensor product of matrices is introduced and some new properties are obtained. Using this tool, a formula is obtained, which realizes Poincare's formal change of coordinates. Finally, this result is used to get conditions for non-regular state feedback (approximate) linearization.

The paper is organized as follows. In Section 2, based on a new feedback canonical form a recursive condition for single-input linearization is obtained. In Section 3 we introduce the concepts and some properties of left semi-tensor product of matrices, which provide a numerical tool for computing normal form, etc. Section 4 gives a formula for Poincare's formal change of coordinates. Then the conditions for non-regular state feedback (approximate) linearization are presented. Section 5 is an illustrative example.

2. Single-input linearization

We first consider a canonical form of non-regular feedback linear systems.

A constant vector, $b = (b_1, \ldots, b_n)^{\mathrm{T}} \in \mathbb{R}^n$, is said to be of non-zero component if $b_i \neq 0$, $\forall i$.

Proposition 2.1. *A linear control system*

$$\dot{x} = Ax + \sum_{i=1}^m b_i u_i := Ax + Bu, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \tag{6}$$

is completely controllable, iff there exists two matrices F, G such that the closed-loop system

$$\dot{x} = (A + BF)x + BGv$$

can be converted, by a linear coordinate change, into the following form:

$$\dot{z} = Az + bv := \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & d_n \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} v, \tag{7}$$

where d_i, $i = 1, \ldots, n$ are distinct and b is of non-zero component.

The proof is straightforward. The key is that the controllability matrix C of such a system is a mild variation of Vandermonde's matrix

$$\det(C) = \prod_{i=1}^n b_i \prod_{i<j} (d_j - d_i) \neq 0. \tag{8}$$

From the above proposition we can call (7) the reduced single-input feedback A-diagonal (RSIFAD) canonical form. Moreover, we give the following assumption:

A1. A is a diagonal matrix with distinct diagonal elements d_i and is non-resonant.

Lemma 2.2. *Assume matrix A satisfies* A1 *and g is a kth degree homogeneous vector field, $k \geqslant 2$. Then there exists a kth degree homogeneous vector field η such that*

$$ad_{Ax}\eta = g. \tag{9}$$

Proof. For a given η, let $f = ad_{Ax}\eta$. Then a straightforward computation shows that f_i, the ith component of f depends only on η_i, the ith component of η. Now let $x_1^{r_1}\cdots x_n^{r_n}$ be a term of η_i, then a straightforward computation shows that

$$ad_{Ax}\eta = \left[\begin{pmatrix} d_1x_1 \\ \cdot \\ d_ix_i \\ \cdot \\ d_nx_n \end{pmatrix}, \begin{pmatrix} \times \\ \cdot \\ x_1^{r_1}\cdot x_n^{r_n} \\ \cdot \\ \times \end{pmatrix}\right] = \begin{pmatrix} \times \\ \cdot \\ \mu_i x_1^{r_1}\cdot x_n^{r_n} \\ \cdot \\ \times \end{pmatrix}, \tag{10}$$

where

$$\mu_i = d_1r_1 + \cdots + d_nr_n - d_i, \tag{11}$$

since A is non-resonant and $\mu_i \neq 0$. Now for each term $x_1^{r_1}\cdots x_n^{r_n}$ of g_i we can construct a corresponding term $1/\mu_i x_1^{r_1}\cdots x_n^{r_n}$ of η_i such that $ad_{Ax}\eta = g$. □

Since all the vector fields and functions are assumed to be analytic, all the functions and their derivatives have convergent Taylor series expansions.

Note that if A satisfies A1, then for a vector field $g = g_kx^k + g_{k+1}x^{k+1} + \cdots \in 0(\|x\|^k)$, applying Lemma 2.2 to each term, we can find a vector field $\eta \in 0(\|x\|^k)$ such that $ad_{Ax}\eta = g$.

Now let us get back to the linearization. We consider the following system:

$$\dot{x} = Ax + \xi(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{12}$$

where A satisfies A1 and $\xi(x) = 0(\|x\|^2)$. An immediate result is

Proposition 2.3. *Consider system* (12) *with A satisfying* A1. *It is non-regular state feedback linearizable if* 1. $\xi(x) \in Span\{g_1,\ldots,g_m\}$; 2. *There exists a constant vector b of non-zero component*, *such that*

$b \in Span\{g_1,\ldots,g_m\}$.

When either of the conditions of Proposition 2.3 fails, we can use the normal form to further investigate the problem.

According to Lemma 2.2, we can always find a vector field $\eta(x)$ such that

$$ad_{Ax}\eta(x) = \xi(x). \tag{13}$$

Now we define a local diffeomorphism as $z_1 = x - \eta(x)$. Then under the coordinate frame z_1 system (12) can be expressed as

$$\dot{z}_1 = Az_1 - J_0(x)\xi(x) + \sum_{i=1}^{m} g_i^1(x)u_i, \tag{14}$$

where $J_0(x)$ is the Jacobian matrix of $\eta(x)$ and $g_i^1(x) = (I - J_0(x))g_i(x)$.

For notational ease, we denote $x := z_0$, $\xi(x) := \xi_0(x)$, $\eta(x) := \eta_0(x)$ and $g_i(x) := g_i^0(x)$. Then we can continue the previous procedure to define recursively new coordinates as

$$ad_{Ax}(\eta_k) = \xi_k, \quad z_{k+1} = z_k - \eta_k(x), \quad k \geqslant 0$$

and new vector fields

$$g_i^{k+1}(x) = (I - J_k(x))g_i^k(x), \quad 1 \leqslant i \leqslant m, \quad k \geqslant 0,$$

where $J_k(x)$ is the Jacobian matrix of $\eta_k(x)$. Then under z_k the system is expressed as

$$\dot{z}_k = Az_k + \xi_k(x) + \sum_{i=1}^{m} g_i^k(x)u_i, \quad k \geqslant 1. \tag{15}$$

As the consequence of the above discussion, we have

Corollary 2.4. *System* (12) *is non-regular state feedback linearizable if there exists $k \geqslant 0$, such that* (15) *satisfies the Conditions* 1 *and* 2 *of Proposition* 2.3.

Note that according to the recursive algorithm it is easy to see that

$$\deg(\xi_i) = c_{i+1} + 1, \quad i = 0, 1, \ldots,$$

where $\{c_i\}$ is the Fibonacci sequence, i.e., $(c_1, c_2, \ldots) = (1, 1, 2, 3, 5, 8, \ldots)$. Hence when $k \to \infty$ $\xi_k(x) \to 0$, because the convergence is assumed. Then we have the following:

Corollary 2.5. *System* (12) *is non-regular state feedback linearizable if there exists a constant vector b of non-zero component such that*

$$b \in Span\left\{\prod_{i=0}^{\infty}(I - J_i(x))g_j(x),\ j = 1,\ldots,m\right\}.$$

3. Left semi-tensor product of matrices

Propositions 2.3, Corollary 2.4 and Proposition 2.5 are perhaps theoretically interesting. However, the algorithm in the last section is recursive. To the authors' best knowledge, all known normal form algorithms are recursive. To get a non-recursive linearization algorithm a new matrix product, called the left semi-tensor product (Cheng, 2002), is studied in this section.

Definition 3.1. (1) *Let X be a row vector of dimension np, and Y be a column vector with dimension p. Then we split X into p equal-size blocks as $X^1,\ldots,X^p$, which are $1\times n$ rows. Define the left semi-tensor product*, *denoted by* $\vdash$, *as*

$$X \ltimes Y = \sum_{i=1}^{p} X^i y_i \in \mathbb{R}^n,$$

$$Y^{\mathrm{T}} \ltimes X^{\mathrm{T}} = \sum_{i=1}^{p} y_i (X^i)^{\mathrm{T}} \in \mathbb{R}^n. \tag{16}$$

(2) Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. If either n is a factor of p, say $nt = p$ and denote it as $A \prec_t B$, or p is a factor of n, say $n = pt$ and denote it as $A \succ_t B$, then we define the left semi-tensor product of A and B , denoted by $C = A \ltimes B$, as

the following: C consists of $m \times q$ blocks as $C = (C^{ij})$ and each block is

$$C^{ij} = A^i \ltimes B_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, q,$$

where A^i is the ith row of A and B_j is the jth column of B.

Example 3.2. 1. Let $X = (1\ 2\ 3\ -1)$ and $Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then $X \ltimes Y = (1\ 2) \cdot 1 + (3\ -1) \cdot 2 = (7\ 0)$.

2. Calculate the product of

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

$$A \ltimes B = \begin{pmatrix} (3\ 4) & (-3\ -5) \\ (4\ 7) & (-5\ -8) \\ (5\ 2) & (-7\ -4) \end{pmatrix} = \begin{pmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{pmatrix}.$$

Remark 3.3. Note that when $n = p$ the left semi-tensor product coincides with the conventional matrix product. Therefore, the left semi-tensor product is only a generalization of the conventional product. For convenience, we may omit the product symbol $\ltimes$.

Some related fundamental properties are:

Proposition 3.4 (Cheng, 2002). *The left semi-tensor product satisfies* (*as long as the related products are well defined*)

1. (*Distributive rule*)

$$A \ltimes (\alpha B + \beta C) = \alpha A \ltimes B + \beta A \ltimes C,$$

$$(\alpha B + \beta C) \ltimes A = \alpha B \ltimes A + \beta C \ltimes A, \quad \alpha, \beta \in \mathbb{R}. \tag{17}$$

2. (*Associative rule*)

$$A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C,$$

$$(B \ltimes C) \ltimes A = B \ltimes (C \ltimes A). \tag{18}$$

The following three propositions can be proved by straightforward computations.

Proposition 3.5. *Let $A \in M_{p\times q}$ and $B \in M_{m\times n}$. If $q = km$, then*

$$A \ltimes B = A(B \otimes I_k), \tag{19}$$

if $kq = m$, then

$$A \ltimes B = (A \otimes I_k)B. \tag{20}$$

Proposition 3.6. 1. *Assume A and B are of the proper dimensions such that $A \ltimes B$ is well defined. Then*

$$(A \ltimes B)^{\mathrm{T}} = B^{\mathrm{T}} \ltimes A^{\mathrm{T}}. \tag{21}$$

2. *In addition assume both A and B are invertible, then*

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{22}$$

Proposition 3.7. *Assume $A \in M_{m\times n}$ is given.* 1. *Let $Z \in \mathbb{R}^t$ be a row vector. Then*

$$A \ltimes Z = Z \ltimes (I_t \otimes A); \tag{23}$$

2. *Let $Z \in \mathbb{R}^t$ be a column vector. Then*

$$Z \ltimes A = (I_t \otimes A) \ltimes Z. \tag{24}$$

Note that when $\xi \in \mathbb{R}^n$ is a column or a row, then

$$\xi^k := \underbrace{\xi \ltimes \cdots \ltimes \xi}_{k}$$

is well defined.

Next, we define a *swap matrix*, which is also called a permutation matrix and is defined implicitly in Magnus and Neudecker (1999). Many properties can be found in Cheng (2002). The swap matrix, $W_{[m,n]}$ is an $mn \times mn$ matrix constructed in the following way: label its columns by $(11, 12, \ldots, 1n, \ldots, m1, m2, \ldots, mn)$ and its rows by $(11, 21, \ldots, m1, \ldots, 1n, 2n, \ldots, mn)$. Then its element in the position $((I,J),(i,j))$ is assigned as

$$w_{(IJ),(ij)} = \delta_{i,j}^{I,J} = \begin{cases} 1 & I = i \text{ and } J = j, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

When $m = n$ we simply denote by $W_{[n]}$ for $W_{[n,n]}$.

Example 3.8. Let $m = 2$ and $n = 3$, the swap matrix $W_{[2,3]}$ is constructed as

$$W_{[2,3]} = \begin{array}{c} \begin{matrix} (11) & (12) & (13) & (21) & (22) & (23) \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{matrix} (11) \\ (21) \\ (12) \\ (22) \\ (13) \\ (23) \end{matrix}.$$

As a consequence of the definition, the following "swap" property is useful in the sequel.

Proposition 3.9. *Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then*

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X,$$

$$W_{[n,m]} \ltimes Y \ltimes X = X \ltimes Y. \tag{26}$$

Proposition 3.10. *Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. Then*

$$W_{[m,p]}(A \otimes B)W_{[q,n]} = B \otimes A. \tag{27}$$

Proof. Applying Propositions 3.7 and 3.9 to each row and/or column, the result follows from a careful computation. □

The following factorization formula is useful for simplifying swap matrix computation.

Proposition 3.11.

$$W_{[pq,r]} = (W_{[p,r]} \otimes I_q)(I_p \otimes W_{[q,r]}). \tag{28}$$

See item 1 of the Appendix for the proof.

For our later results, we need also to introduce the tensor expression of polynomials. Let $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. Then a kth degree homogeneous polynomial can be expressed as $F \ltimes x^k$, where

$$x^k := \underbrace{x \ltimes \cdots \ltimes x}_{k},$$

and the coefficient vector F is a $1 \times n^k$ row. Briefly, $Fx^k := F \ltimes x^k$. By definition we have

Proposition 3.12. *Let Fx^m and Gx^n be mth and nth homogeneous polynomials respectively. Then the product is*

$$(Fx^m)(Gx^n) = F \ltimes G \ltimes x^{m+n}. \tag{29}$$

Next, we consider the differential of a matrix with differentiable function entries.

Definition 3.13. Let $H = (h_{ij}(x))$ be a $p \times q$ matrix with the entries $h_{ij}(x)$ as smooth functions of $x \in \mathbb{R}^n$. Then the differential of H is defined as a $p \times nq$ matrix obtained by replacing each element h_{ij} by its differential

$$\mathrm{d}h_{ij} = \left(\frac{\partial h_{ij}(x)}{\partial x_1}, \cdots, \frac{\partial h_{ij}(x)}{\partial x_n} \right).$$

The following formula is fundamental:

Proposition 3.14.

$$D(x^{k+1}) = \Phi_k \ltimes x^k, \tag{30}$$

where Φ_k is an $n^{k+1} \times n^{k+1}$ matrix as

$$\Phi_k = \sum_{s=0}^{k} I_{n^s} \otimes W_{[n^{k-s},n]}. \tag{31}$$

See item 2 of the Appendix for the proof.

The higher degree differential is defined inductively as

$$D^{k+1}A(x) = D(D^k A(x)), \quad k \geqslant 1.$$

Using the above expression, the Taylor expression of a vector field $f(x) \in V(\mathbb{R}^n)$ is expressed as

$$f(x) = f(x_0) + \sum_{i=1}^{\infty} \frac{1}{i!} D^i f(x_0)(x - x_0)^i, \tag{32}$$

which has exactly the same form as the Taylor expansion of one variable case.

4. Algorithm for (approximate) linearization

In this section we present a formula to realize Poincare's coordinate transformation (4) first. Then we give necessary and sufficient conditions for (approximate) linearizability.

To begin with, using Taylor series expression on $f(x)$ with the form of semi-tensor product, we express system (3) as

$$\dot{x} = Ax + F_2 x^2 + F_3 x^3 + \cdots, \tag{33}$$

where F_k are $n \times n^k$ constant matrices, and x^k are as stated above.

Next, assume

$$ad_{Ax}\eta_k = F_k x^k.$$

Using Lemma 2.2, we can easily obtain that

$$\eta_k = (\Gamma_k^n \odot F_k)x^k, \quad x \in \mathbb{R}^n. \tag{34}$$

Here $\odot$ is the Hadamard product of matrices (Zhang, 1999), i.e., if two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ are given, and $C = A \odot B$, then C is of the same size with entries $c_{ij} = a_{ij} b_{ij}$. Γ_k^n can be constructed by (11) mechanically as

$$(\Gamma_k^n)_{ij} = \frac{1}{(\sum_{s=1}^{n} \alpha_s^j \lambda_s) - \lambda_i}, \quad i = 1, \dots, n; \; j = 1, \dots, n^k, \tag{35}$$

where $\alpha_1^j, \dots, \alpha_n^j$ are respectively the powers of $x_1, \dots, x_n$ of the jth component of x^k.

Now we are ready to present our main result:

Theorem 4.1. *Assume A satisfies $A1$. Then system (33) can be transformed into a linear form*

$$\dot{z} = Az \tag{36}$$

by the following coordinate transformation:

$$z = x - \sum_{i=2}^{\infty} E_i x^i, \tag{37}$$

where E_i are determined recursively as (with Φ_i as in (31))

$$E_2 = \Gamma_2 \odot F_2$$

$$E_s = \Gamma_s \odot \left(F_s - \sum_{i=2}^{s-1} E_i \Phi_{i-1} (I_{n^{i-1}} \otimes F_{s+1-i}) \right),$$

$$s \geqslant 3. \tag{38}$$

Proof. Using (1) to system (33), we have

$$\dot{z} = \left(Ax + \sum_{i=2}^{\infty} F_i x^i \right) - \sum_{i=2}^{\infty} \frac{\partial E_i x^i}{\partial x} \left(Ax + \sum_{i=2}^{\infty} F_i x^i \right)$$

$$= Az + \sum_{i=2}^{\infty} F_i x^i + A \sum_{i=2}^{\infty} E_i x^i - \sum_{i=2}^{\infty} \frac{\partial E_i x^i}{\partial x} Ax$$

$$- \left(\sum_{i=2}^{\infty} \frac{\partial E_i x^i}{\partial x} \right) \left(\sum_{j=2}^{\infty} F_j x^j \right)$$

$$= Az - \sum_{i=2}^{\infty} ad_{Ax}(E_i x^i) + F_2 x^2$$

$$+\sum_{s=3}^{\infty}\left(F_s x^s - \sum_{i=2}^{s-1}\frac{\partial E_i x^i}{\partial x} F_{s+1-i} x^{s+1-i}\right)$$

$$:= Az - \sum_{i=2}^{\infty} ad_{Ax}(E_i x^i) + \sum_{s=2}^{\infty} L_s, \tag{39}$$

where

$$L_2 = F_2 x^2,$$

$$L_s = F_s x^s - \sum_{i=2}^{s-1}\frac{\partial E_i x^i}{\partial x} \ltimes F_{s+1-i} x^{s+1-i}$$

$$= \left(F_s - \sum_{i=2}^{s-1} E_i \Phi_{i-1}(I_{n^{i-1}} \otimes F_{s+1-i})\right) x^s, \ s \geqslant 3. \tag{40}$$

Now under Assumption A1 we can set

$$E_s x^s = ad_{Ax}^{-1}(L_s), \quad s = 2, 3, \ldots .$$

Then (33) becomes (4). □

The advantage of this Taylor series expression is that it does not require recursive computation of the intermediate forms of the system under transferring coordinates z_i, $i = 1, 2, 3, \ldots$.

Now consider the linearization of system (1). Denote $A = (\partial f/\partial x)|_0$, $B = g(0)$, and assume (A, B) is a completely controllable pair. Then we can find feedback K and a linear coordinate transformation T, such that $\tilde{A} = T^{-1}(A + BK)T$ satisfies assumption A1. For the sake of simplicity, we call the above transformation an *NR-type transformation*.

Using the notations and algorithm proposed above the following result is immediate.

Theorem 4.2. *System* (1) *is single-input linearizable, iff there exist an NR-type transformation and a constant vector b of non-zero component such that*

$$b \in Span\left\{\left(I - \sum_{i=2}^{\infty} E_i \Phi_{i-1} x^{i-1}\right) g_j \mid j = 1, \ldots, m\right\}. \tag{41}$$

Next, we consider approximate linearization.

Definition 4.3. System (1) is said to be kth degree non-regular state feedback approximately linearizable if we can find a state feedback and a local coordinate frame z such that under z the closed-loop system can be expressed as

$$\dot{z} = Az + 0(\|z\|^{k+1}) + (b + 0(\|z\|^k))v, \tag{42}$$

where (A, b) is a completely controllable pair.

For approximate linearization, we may relax the non-resonant constraint.

Definition 4.4. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of a given matrix A. A is a kth degree resonant matrix if there exists $m = (m_1, \ldots, m_n) \in Z_+^n$, and $2 \leqslant |m| \leqslant k$, such that for some s, $\lambda_s = \langle m, \lambda \rangle$.

From the expression (35) it is clear that the following corollary of the Poincare's Lemma is correct.

Corollary 4.5. *Consider a* C^ω *dynamic system* (3). *If A is kth degree non-resonant, there exists a formal change of coordinates* (4), *such that system* (3) *can be expressed as*

$$\dot{z} = Az + 0(\|z\|^{k+1}). \tag{43}$$

If we just consider the kth degree approximate linearization of system (33) with $u_i = 0$, we need to adjust (A.1) as

$$z = x - \sum_{i=2}^{k} E_i x^i. \tag{44}$$

In this case, (A.5) is still valid (here $s \leqslant k$), and (A.4) will become

$$\dot{z} = Az + 0(\|x\|^{k+1}). \tag{45}$$

We say a transformation is NR-k-*type transformation*, if it is almost the same as NR-type transformation except the non-resonant condition, which is replaced by the kth degree non-resonance.

Theorem 4.6. *System* (1) *is kth degree single-input approximate state feedback linearizable, iff there exist an NR-k-type transformation and a constant vector b of non-zero component such that*

$$b \in Span\left\{\left(I - \sum_{i=2}^{k} E_i \Phi_{i-1} x^{i-1}\right) g_j \mid \forall j\right\}$$
$$+ 0(\|x\|^k). \tag{46}$$

5. An illustrative example

We use the following example to demonstrate the linearizing process.

Example 5.1. Find a 4th degree approximate state feedback linearization of the following control system:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix} -4\sin x_1 - \frac{2}{3}x_1^3 + 5x_2^2 + 6x_2^3\\ -5x_2 - 3x_3^2\\ -6x_3\end{pmatrix} + \begin{pmatrix}0\\ 6(1+x_3)\\ 7\end{pmatrix} u_1 + \begin{pmatrix}1\\0\\0\end{pmatrix} u_2. \tag{47}$$

Using Taylor series expansion, we express (47) as

$$\dot{x} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{pmatrix} x + \begin{pmatrix} 5x_2^2 \\ -3x_3^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 6x_2^3 \\ 0 \\ 0 \end{pmatrix}$$

$$+ 0(\|x\|^5) + \begin{pmatrix} 0 \\ 6+6x_3 \\ 7 \end{pmatrix} u_1 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_2. \qquad (48)$$

It is easy to calculate that

$$L_2 = (5x_2^2,\ -3x_3^2,\ 0)^{\mathrm{T}},$$

$$E_2x^2 = ad_{Ax}^{-1}(L_2) = \left(-\frac{5}{6}x_2^2,\ \frac{3}{7}x_3^2,\ 0\right)^{\mathrm{T}},$$

$$L_3 = (6x_2^3 - 5x_2x_3^2,\ 0,\ 0)^{\mathrm{T}},$$

$$E_3x^3 = ad_{Ax}^{-1}(L_3) = \left(-\frac{6}{11}x_2^3 + \frac{5}{13}x_2x_3^2,\ 0,\ 0\right)^{\mathrm{T}}.$$

So we get the desired coordinate transformation as follows:

$$z = x - \begin{pmatrix} -\frac{5}{6}x_2^2 - \frac{6}{11}x_2^3 + \frac{5}{13}x_2x_3^2 \\ \frac{3}{7}x_3^2 \\ 0 \end{pmatrix}, \qquad (49)$$

under which system (47) is expressed as

$$\dot{z} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{pmatrix} z + 0(\|x\|^5)$$

$$+ \begin{pmatrix} h(x) & 1 \\ 6 & 0 \\ 7 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad (50)$$

where $h(x) = (6+6x_3)(\frac{5}{3}x_2 + \frac{18}{11}x_2^2 - \frac{5}{13}x_3^2) - \frac{70}{13}x_2x_3$. Because

$$\begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} h(x) \\ 6 \\ 7 \end{pmatrix} \times 1 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times (-h(x)+1),$$

Theorem 4.6 assures that the system is 4th degree approximate state feedback linearizable.

Choosing state feedback

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -h(x)+1 \end{pmatrix} v \qquad (51)$$

and substituting it into (50), we get

$$z = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{pmatrix} z + 0(\|x\|^5) + \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix} v, \qquad (52)$$

which is the desired 4th degree approximate state feedback linearization of system (47).

6. Conclusion

In this paper, non-regular state feedback (approximate) linearization was investigated by using the normal form theory. First, we proposed a new canonical form for non-regular feedback linear control systems. Based on it, a set of recursive conditions were obtained for the linearization. Then we introduced the left semi-tensor product of matrices and developed some properties. Implementing it a formula was obtained to realize Poincare's formal change of coordinates. In light of this result, certain conditions were revealed for the non-regular state feedback (approximate) linearization.

Appendix A.

A.1. *Proof of Proposition 3.11*

Let $X \in \mathbb{R}^p$, $Y \in \mathbb{R}^q$ and $Z \in \mathbb{R}^r$ be column vectors. By definition it is easy to see that for any two column vectors

$$X \ltimes Y = X \otimes Y.$$

It follows from Proposition 3.9 that

$$\begin{aligned}(I_p \otimes W_{[q,r]})(X \ltimes Y \ltimes Z) &= (I_p \otimes W_{[q,r]})(X \otimes (Y \ltimes Z)) \\ &= X \otimes (W_{[q,r]}Y \ltimes Z) \\ &= X \otimes (Z \ltimes Y) = X \ltimes Z \ltimes Y.\end{aligned}$$

Similarly,

$$(W_{[p,r]} \otimes I_q)(X \ltimes Z \ltimes Y) = Z \ltimes X \ltimes Y.$$

It follows that for any $X \in \mathbb{R}^p$, $Y \in \mathbb{R}^q$ and $Z \in \mathbb{R}^r$ we have

$$W_{[pq,r]}(X \ltimes Y \ltimes Z) = (W_{[p,r]} \otimes I_q)$$

$$(I_p \otimes W_{[q,r]})(X \ltimes Y \ltimes Z) = Z \ltimes X \ltimes Y. \qquad (A.1)$$

Note that $\{X \ltimes Y \ltimes Z | X \in \mathbb{R}^p, Y \in \mathbb{R}^q, \text{ and } Z \in \mathbb{R}^r\}$ is not a vector space, but it contains a basis of $\mathbb{R}^{p+q+r}$, which is generated by $\delta_i \ltimes \delta_j \ltimes \delta_k$. So (A.1) implies (28).

A.2. Proof of Proposition 3.14

We sketch the proof in several steps. First lemma follows from a straightforward computation.

Lemma A.1. *For conventional matrix product $A(x)B(x)$ we have*

$$D(A(x)B(x)) = DA(x) \ltimes B(x) + A(x) \ltimes DB(x). \tag{A.2}$$

Lemma A.2. *Let $A(x) \in M_{p\times q}$, with $x \in R^n$. Then*

$$D(A \otimes I_k) = (DA \otimes I_k)(I_q \otimes W_{[k,n]}). \tag{A.3}$$

Proof. It is easy to verify that both $D(A(x)\otimes I_k)$ and $DA\otimes I_k$ can be split into $p \times q$ blocks. Denote them by $D(A(x) \otimes I_k) = E(x) = \{E_{ij}(x)\}$ and $DA(x) \otimes I_k = F(x) = \{F_{ij}(x)\}$ respectively. Then

$$E_{ij}(x) = D(a_{ij}(x)I_k), \quad F_{ij}(x) = da_{ij}(x) \otimes I_k.$$

A straightforward computation shows that if we index the columns of $F_{ij}(x)$ by (ij) in the order of $i - j$, that is, j goes from 1 to k first then i goes from 1 to n. Precisely, the order is $(11, 12, \dots, 1, k, \dots, n1, n2, \dots, n, k)$, while E_{ij} consists of the same columns only in order of $j - i$. According to Proposition 3.9, $E_{ij} = F_{ij}W_{[k,n]}$. Now set

$$W = \operatorname{diag}\left(\underbrace{W_{[k,n]}, \dots, W_{[k,n]}}_{q}\right),$$

then $E = FW$. (A.3) follows. □

Lemma A.3. *Let $A(x) \in M_{p\times q}$ and $B(x) \in M_{tq\times s}$, where $x \in R^n$. Then*

$$D(A(x) \ltimes B(x)) = DA(x) \ltimes (I_q \otimes W_{[t,n]}) \ltimes B(x) + A \ltimes DB(x). \tag{A.4}$$

Proof. Using Proposition 3.5, $A(x)\ltimes B(x) = (A(x)\otimes I_t)B(x)$. Using Lemmas A.2 and A.3, we have

$$\begin{aligned} D(A(x) \ltimes B(x)) &= D[(A(x) \otimes I_t)B(x)] \\ &= (DA \otimes I_t)(I_q \otimes W_{[t,n]}) \ltimes B(x) \\ &\quad + A(x) \ltimes DB(x). \end{aligned}$$

Then (A.4) follows. □

Lemma A.4.

$$D(x^k) = \sum_{i=1}^{k-1} x^{i-1} W_{[n^{k-i},n]} x^{k-i} + x^{k-1} \otimes I_n, \ k \geqslant 2. \tag{A.5}$$

Proof. We prove it by mathematical induction. It is clear that

$$Dx = I_n.$$

Using Lemma A.3, we have

$$\begin{aligned} D(x^2) &= Dx \ltimes (1 \otimes W_{[n]}) \ltimes x + x \ltimes I_n \\ &= I_n \ltimes W_{[n]} \ltimes x + (x \otimes I_n)I_n = W_{[n]} \ltimes x + x \otimes I_n. \end{aligned}$$

Assume (A.5) holds for k. Using Lemma A.3 again, we have

$$D(x \otimes I_{n^k}) = (I_n \otimes I_{n^k})(1 \otimes W_{[n^k,n]}) = W_{[n^k,n]}.$$

Then

$$\begin{aligned} D(x^{k+1}) &= D[(x \otimes I_{n^k})x^k] \\ &= D(x \otimes I_{n^k})x^k + (x \otimes I_{n^k})D(x^k) \\ &= W_{[n^k,n]}x^k + (x \otimes I_{n^k})(W_{[n^{k-1},n]}x^{k-1} \\ &\quad + xW_{[n^{k-2},n]}x^{k-2} + \cdots + x^{k-1} \otimes I_n) \\ &= W_{[n^k,n]}x^k + xW_{[n^{k-1},n]}x^{k-1} + \cdots + x^k \otimes I_n. \quad \square \end{aligned}$$

Lemma A.5.

$$x^p W_{[n^s,n]} = (I_{n^p} \otimes W_{[n^s,n]})x^p. \tag{A.6}$$

Proof. Using Proposition 3.7, the result follows. □

Proof of Proposition 4.13. Using (A.6) to each term of (A.5), Eq. (30) follows immediately. □

References

Arnold, V. I. (1983). *Geometrical methods on the theory of ordinary differential equations* (pp. 177–188). New York: Springer.

Cheng, D. (2002). *Matrix and polynomial approach to dynamic control systems*. Beijing: Science Press.

Cheng, D., & Martin, C. (2002). Normal form representation of control systems. *International Journal of Robust and Nonlinear Control*, *12*, 409–433.

Desoer, C. A., & Vidyasagar, M. (1975). *Feedback systems: input–output properties*. New York: Academic Press.

Devanathan, R. (2001). Linearization condition through state feedback. *IEEE Transactions on Automatic Control*, *46*(8), 1257–1260.

Ge, S. S., Sun, Z., & Lee, T. H. (2001). Nonregular feedback linearization for a class of second-order nonlinear systems. *Automatica*, *37*, 1819–1824.

Guay, M. (2002). On the linearizability of nonisothermal continuous stirred-tank reactors. *Automatica*, *38*(2), 269–278.

Guckenheimer, J., & Holmes, P. (1983). *Nonlinear oscillations. Dynamical systems and bifurcations of vector fields*. Berlin: Springer.

Heymann, M. (1968). Pole assignment in multi-input linear systems. *IEEE Transactions on Automatic Control*, *13*(6), 748–749.

Krener, A. J., & Kang, W. (1990). Extended normal forms of quadratic systems. *Proceedings of the 29th Conference on Decision Control* (pp. 2091–2095).

Krener, A. J., Karahan, S., & Hubbard, M. (1988). Approximate normal forms of nonlinear systems. *Proceedings of the 27th Conference on Decision Control* (pp. 1223–1229).

Magnus, J. R., & Neudecker, H. (1999). *Matrix differential calculus with applications in statistics and econometrics* (Revised Ed.). New York: Wiley.

Sun, Z., & Xia, X. (1997). On nonregular feedback linearization. *Automatica*, *33*(7), 1339–1344.

Zhang, F. (1999). *Matrix theory. Basic results and techniques*. Berlin: Springer.

Daizhan Cheng received Ph.D. degree from Washington University, MO, in 1985. Since 1990, he has been a Professor with the Institute of Systems Science. Chinese Academy of Sciences. He was an Associate Editor of Math Sys., Est. Contr. (91-93), and Automatica (98-02). He is an Associate Editor of Asia J. Control., Deputy Chief Editor of Control and Decision, J. Control Theory and App. etc. He is Chairman of the Technical Committee on Control Theory, Chinese Automation Association. His research interests include nonlinear system and control, Numerical Method etc.

Xiaoming Hu was born in Chengdu, China, in 1961. He received the B.S. degree from University of Science and Technology of China in 1983. He received the M.S. and Ph.D degrees from Arizona State University in 1986 and 1989 respectively. From 1989 to 1990 he was a Gustafson Postdoctoral Fellow at the Royal Institute of Technology, Stockholm, where he is currently an associate professor. His main research interests are in nonlinear feedback stabilization, nonlinear observer design, sensing and active perception, motion planning and control of mobile robots, and mobile manipulation.

Yuzhen Wang graduated from Tai'an Teachers College in 1986, received his M.S. degree from Shandong University of Science & Technology in 1995 and his Ph.D. degree from the Institute of Systems Science, Chinese Academy of Sciences in 2001. Now he is a professor with the School of Control Science and Engineering, Shandong University, Jinan, China. His research interests include nonlinear control systems, Hamiltonian systems and robust control. Dr. Wang received the Prize of Guan Zhaozhi in 2002, and the Prize of Huawei from the Chinese Academy of Sciences in 2001.