Chapter 16
Some Applications to Differential Geometry and Algebra

This chapter present some applications of semi-tensor product to Differential Geometry and Algebra. In many mathematical problems we need to deal with multiple dimensional data,or data labeled by multi-index. In this case semi-tensor product could be a proper and useful tool for formula deduction or numerical calculation. First, we consider the calculation of connection, curvature, and Riemann curvature tensor etc. Then we consider the structue of finite dimensional algebra. A set of algebraic equations are obtained to describe finite dimensional Lie algebra. Finally, we prove a formula for tensor contraction, which comes from Relativity.

16.1 Calculation of Connection

Connection is a fundamental concept is Geometry as well as in Physics. In this section we will consider some connection related calculations using semi-tensor product. First, we give the definition [2]

Definition 16.1. Let $f,g \in V(M)$ be two smooth vector fields on a smooth manifold M. An $\mathbb{R}$-bilinear mapping ∇: $V(M) \times V(M) \to V(M)$ is called a connection, if it satisfies

1.
$$\nabla_{rf} sg = rs\nabla_f g, \quad r,s \in \mathbb{R}; \tag{16.1}$$

2.
$$\nabla_{hf} g = h\nabla_f g, \quad \nabla_f (hg) = L_f(h)g + h\nabla_f g, \quad h \in C^\infty(M). \tag{16.2}$$

Note that in this chapter "smooth" means infinitely differentiable, denoted by C^∞.

According to the definition, it is easy to verify that if a connection is defined over a basis of $V(M)$, is is overall well defined. Hence, we need to define the action of connection over basis vectors. Under a local coordinate chart x, the action of a connection is determined by the following forms.

$$\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right)=\sum_{k=1}^{n}\gamma_{ij}^{k}\frac{\partial}{\partial x_k},$$

where γ_{ij}^{k} is called the Christoffel symbol. Using Christoffel symbol, we can construct a matrix

$$\Gamma=\begin{bmatrix}\gamma_{11}^{1} & \cdots & \gamma_{1n}^{1} & \cdots & \gamma_{n1}^{1} & \cdots & \gamma_{nn}^{1}\\ \vdots & & \vdots & & \vdots & & \vdots\\ \gamma_{11}^{n} & \cdots & \gamma_{1n}^{n} & \cdots & \gamma_{n1}^{n} & \cdots & \gamma_{nn}^{n}\end{bmatrix},$$

which is called the Christoffel matrix.

Next, we give a matrix form of the connection.

Proposition 16.1. *Let* $f=\sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i}$, $g=\sum_{j=1}^{n} g_j\frac{\partial}{\partial x_j}$, *and denote their vector form as* $f=(f_1,f_2,\cdots,f_n)^{\mathrm{T}}$, $g=(g_1,g_2,\cdots,g_n)^{\mathrm{T}}$. *Then*

$$\nabla_f g = Dgf+\Gamma fg. \tag{16.3}$$

Proof. According to the definition (16.1)–(16.2), we can calculate that

$$\begin{aligned}\nabla_f g &= \sum_{i=1}^{n} f_i\left[\sum_{j=1}^{n} L_{\frac{\partial}{\partial x_i}} g_j\frac{\partial}{\partial x_j}+\sum_{j=1}^{n}\sum_{k=1}^{n} g_j\gamma_{ij}^{k}\frac{\partial}{\partial x_k}\right]\\ &= Dg\ltimes f+\Gamma\ltimes f\ltimes g.\end{aligned} \tag{16.4}$$

Then (16.3) follows immediately. □

Now assume $y=y(x)$ is another local coordinate chart, we intend to present the Chrostoffel matrix Γ under this new coordinate frame. Denote by $\tilde{\Gamma}$ and $\tilde{\gamma}_{ij}^{k}$ for new Γ and its entries γ_{ij}^{k} respectively, then we have

Lemma 16.1. *Under new coordinate frame y the Chrostoffel matrix can be calculated as*

$$\begin{aligned}\begin{bmatrix}\tilde{\gamma}_{ij}^{1}\\ \vdots\\ \tilde{\gamma}_{ij}^{n}\end{bmatrix} &= \begin{bmatrix}\frac{\partial^2 x_1}{\partial y_j\partial y_1} & \cdots & \frac{\partial^2 x_1}{\partial y_j\partial y_n}\\ \vdots & & \\ \frac{\partial^2 x_n}{\partial y_j\partial y_1} & \cdots & \frac{\partial^2 x_n}{\partial y_j\partial y_n}\end{bmatrix}\left[\frac{\partial x_1}{\partial y_i},\cdots,\frac{\partial x_n}{\partial y_i}\right]^{\mathrm{T}}\\ &\quad+\Gamma\ltimes\left[\frac{\partial x_1}{\partial y_i},\cdots,\frac{\partial x_n}{\partial y_i}\right]^{\mathrm{T}}\ltimes\left[\frac{\partial x_1}{\partial y_j}\cdots\frac{\partial x_n}{\partial y_j}\right]^{\mathrm{T}}.\end{aligned} \tag{16.5}$$

Proof. Set

$$f = \frac{\partial}{\partial y_i} = \sum_{s=1}^{n} \frac{\partial}{\partial x_s} \frac{\partial x_s}{\partial y_i},$$

and

$$g = \frac{\partial}{\partial y_j} = \sum_{t=1}^{n} \frac{\partial}{\partial x_t} \frac{\partial x_t}{\partial y_j}.$$

Recall the definition of γ, we have

$$\sum_{k=1}^{n} \tilde{\gamma}_{ij}^{k} \frac{\partial}{\partial y_k} = \nabla_f g.$$

Applying (16.3) to the above equation, equation (16.5) follows immediately. □

Theorem 16.1. *Under new coordinate frame y we have*

$$\tilde{\Gamma} = D^2 x Dx + \Gamma \ltimes Dx(I \otimes Dx). \tag{16.6}$$

Proof. A direct calculation shows that

$$D^2 x \ltimes Dx = \begin{bmatrix} \sum\limits_{s=1}^{n} \frac{\partial^2 x_1}{\partial y_s \partial y_1} \frac{\partial x_s}{\partial y_1} & \cdots & \sum\limits_{s=1}^{n} \frac{\partial^2 x_1}{\partial y_s \partial y_n} \frac{\partial x_s}{\partial y_1} & \cdots & \sum\limits_{s=1}^{n} \frac{\partial^2 x_1}{\partial y_s \partial y_1} \frac{\partial x_s}{\partial y_n} & \cdots & \sum\limits_{s=1}^{n} \frac{\partial^2 x_1}{\partial y_s \partial y_n} \frac{\partial x_s}{\partial y_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sum\limits_{s=1}^{n} \frac{\partial^2 x_n}{\partial y_s \partial y_1} \frac{\partial x_s}{\partial y_1} & \cdots & \sum\limits_{s=1}^{n} \frac{\partial^2 x_n}{\partial y_s \partial y_n} \frac{\partial x_s}{\partial y_1} & \cdots & \sum\limits_{s=1}^{n} \frac{\partial^2 x_n}{\partial y_s \partial y_1} \frac{\partial x_s}{\partial y_n} & \cdots & \sum\limits_{s=1}^{n} \frac{\partial^2 x_n}{\partial y_s \partial y_n} \frac{\partial x_s}{\partial y_n} \end{bmatrix}.$$

If we label $\mathrm{Col}(D^2 x \ltimes Dx)$ by (ij) in the order of $\mathrm{id}(i,j;n,n)$, then it is easy to verify that its (i,j)-th column is the first term of the right hand side of (16.5).

Denote by $J_i = \mathrm{Col}_i(Dx)$ the i-th column of Dx, then

$$\Gamma \ltimes Dx = (\Gamma \ltimes J_1, \Gamma \ltimes J_2, \cdots, \Gamma \ltimes J_n).$$

We also have $I \otimes Dx = \mathrm{diag}(J, \cdots, J)$, where $J = (J_1, \cdots, J_n)$. Hence

$$\begin{aligned} &\Gamma \ltimes Dx \ltimes (I \otimes Dx) \\ &\quad = (\Gamma \ltimes J_1 \ltimes J_1, \cdots, \Gamma \ltimes J_1 \ltimes J_n, \cdots, \Gamma \ltimes J_n \ltimes J_1, \cdots, \Gamma \ltimes J_n \ltimes J_n). \end{aligned}$$

It is clear that the (ij)-th column of the above, $\mathrm{Col}_{i,j}(\Gamma \ltimes Dx \ltimes (I \otimes Dx))$ is the second term of the right hand side of (16.5). □

Remark 16.1. Using right semi-tensor product, (16.6) can also be expressed as

$$\tilde{\Gamma} = D^2x \ltimes Dx + (\Gamma \ltimes Dx) \rtimes Dx. \tag{16.7}$$

Note that since there is no associativity between left and right semi-tensor product, the parentheses in (16.7) can not be omitted. To avoid possible confusion, we use right semi-tensor product rarely.

Let M be a Riemannian manifold , its Reimannian metric is determined by a positive definite two form, which has its structure matrix $G = (g_{ij})_{n\times n}$. The fundamental theorem of Riemannian Geometry says that, on M there exists an unique connection. Moreover, its Christoffel symbol is determined by the following formula [1].

$$\gamma_{ij}^k = \frac{1}{2}\sum_{s=1}^{n} g^{ks}\left(\frac{\partial g_{si}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_s} + \frac{\partial g_{js}}{\partial x_i}\right), \tag{16.8}$$

where g^{ij} is the (i,j)-th element of G^{-1}.

Using this uniquely determined connection, on a Riemannian manifold we have [2]

$$[f,g] = \nabla_f g - \nabla_g f. \tag{16.9}$$

A Christoffel matrix is said to be symmetric, if

$$\gamma_{ij}^k = \gamma_{ji}^k, \quad \forall i,j,k. \tag{16.10}$$

Then we have the following result.

Theorem 16.2. *If a manifold N has a connection with symmetric Christoffel matrix, then (16.9) holds.*

Proof. It is easy to see that if Christoffel matrix Γ is symmetric, then $\Gamma W_{[n]} = \Gamma$. That is

$$\Gamma fg = \Gamma gf, \quad \forall f,g \in V(N).$$

Using (16.3), we have

$$\nabla_f g - \nabla_g f = Dgf - Dfg = [f,g].$$

□

According to (16.8), it is clear that for a Riemannian manifold the Christoffel matrix is symmetric. Hence (16.9) holds. Therefore, Theorem 16.2 is a more general result.

In both formula deduction and the numerical calculation via computer matrix form is often more convenient. A straightforward computation shows that (16.8) has its matrix form as

$$\Gamma = \frac{1}{2}G^{-1}\left(DG + DGW_{[n]} - (DV_r(G))^{\mathrm{T}}\right). \tag{16.11}$$

A related topic is the geodesic. Let $r(t)$ be a curve on a Riemannian manifold M. $r(t)$ is a geodesic, if and only if [1]

$$\ddot{r}^i = \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma^i_{j,k}\dot{r}^j\dot{r}^k. \tag{16.12}$$

Now under a local coordinate frame $r(t)$ is expressed in its component form as $r(t) = (x_1(t), \cdots, x_n(t))^{\mathrm{T}}$. Then (16.12) has its matrix form as

$$\begin{bmatrix} \ddot{x}_1 \\ \vdots \\ \ddot{x}_n \end{bmatrix} = \Gamma \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix}^2 .$$

Finally, we consider the curvature operator and the Riemannian curvature tensor.

Definition 16.2 ([2]).

1. The curvature operator $R(X,Y)$ is an operator determined by two C^∞ vector fields X and Y, for each C^∞ vector field Z it assigned a C^∞ vector field $R(X,Y)\cdot Z$ as

$$R(X,Y)\cdot Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z. \tag{16.13}$$

2. The Riemannian curvature tensor is a 4-th order C^∞ covariant tensor field defined by

$$\mathscr{R}(X,Y,Z,W) = (R(X,Y)\cdot Z, W). \tag{16.14}$$

We calculate the structure matrix of the curvature operator.

Assume under a local coordinate frame $x = (x_1, \cdots, x_n)$ we have $E_i := \frac{\partial}{\partial x_i}$. Denote the three vector fields in vector form as $X = (\alpha_1, \cdots, \alpha_n)$, $Y = (\beta_1, \cdots, \beta_n)$, and $Z = (\gamma_1, \cdots, \gamma_n)$. Then [2]

$$R(X,Y)\cdot Z = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_i\beta_j\gamma_k R(E_i,E_j)\cdot E_k. \tag{16.15}$$

Since

$$\begin{aligned} \nabla_{E_i}(\nabla_{E_j}E_k) &= \nabla_{E_i}\left(\sum_{t=1}^{n}\gamma^t_{jk}E_t\right) \\ &= \sum_{t=1}^{n} L_{E_i}(\gamma^t_{jk})E_t + \sum_{t=1}^{n}\gamma^t_{jk}\sum_{\ell=1}^{n}\gamma^\ell_{it}E_\ell \\ &= \sum_{t=1}^{n}\left(L_{E_i}(\gamma^t_{jk}) + \sum_{\ell=1}^{n}\gamma^t_{i\ell}\gamma^\ell_{jk}\right)E_t. \end{aligned} \tag{16.16}$$

Similarly,

$$\nabla_{E_j}(\nabla_{E_i}E_k) = \sum_{t=1}^{n}\left(L_{E_j}(\gamma_{ik}^t) + \sum_{\ell=1}^{n}\gamma_{j\ell}^t\gamma_{ik}^\ell\right)E_t. \tag{16.17}$$

Using (16.16) and (16.17) to (16.13) and (16.15), we can construct the structure matrix as follows: Define

$$m_{ijk}^t := \left(L_{E_i}(\gamma_{jk}^t) + \sum_{\ell=1}^{n}\gamma_{i\ell}^t\gamma_{jk}^\ell\right) - \left(L_{E_j}(\gamma_{ik}^t) + \sum_{\ell=1}^{n}\gamma_{j\ell}^t\gamma_{ik}^\ell\right),$$

then the structure matrix of R is

$$M_R = \left(m_{ijk}^t\right), \tag{16.18}$$

where the elements are arranged by $\mathrm{id}(t;n)\times\mathrm{id}(i,j,k;n,n,n)$. It follows that

$$R(X,Y)\cdot Z = M_R XYZ. \tag{16.19}$$

Note that the above deduction is formal. To prove (16.19) we need to show that R is multi-linear with respect to X, Y, and Z. That is, for any smooth functions f, g, and h, we have

$$R(fX, gY, hZ) = fghR(X,Y,Z).$$

In fact, this is correct [2].

Next, we consider the Riemannian curvature tensor. Note that a bilinear form $p^{\mathrm{T}}Gq$ can be equivalently expressed as

$$p^{\mathrm{T}}Gq = \sum_{i,j} g_{ij}p_iq_j = V_r^{\mathrm{T}}(G)pq = V_c^{\mathrm{T}}(G)qp,$$

For the Riemannian curvature tensor observe that

$$\mathscr{R}(X,Y,Z,W) = W^{\mathrm{T}}GR(X,Y)\cdot Z.$$

Using the above bilinear form, we have

$$\mathscr{R}(X,Y,Z,W) = V_c^{\mathrm{T}}(G)M_{\mathscr{R}}XYZW. \tag{16.20}$$

16.2 Structure Analysis of Finite Dimensional Algebra

Definition 16.3 ([6]). An n-dimensional algebra over $\mathbb{R}$ is an n-dimensional vector space $\mathscr{L}$ over $\mathbb{R}$ with a multiplication $*:\mathscr{L}\times\mathscr{L}\to\mathscr{L}$, satisfying distributive rule, that is,

$$\begin{aligned}&(aX+bY)*Z = a(X*Z)+b(Y*Z),\\ &Z*(aX+bY) = a(Z*X)+b(Z*Y), \quad X,Y,Z\in\mathscr{L},\ a,b\in\mathbb{R}.\end{aligned} \tag{16.21}$$

Definition 16.4. Let $\{e_1,\cdots,e_n\}$ be a basis of an algebra $\mathscr{L}$.

1. Assume

$$e_i * e_j = \sum_{k=1}^{n} \alpha_{ij}^k e_k, \quad i,j=1,\cdots,n. \tag{16.22}$$

   Then α_{ij}^k are called the structure constants of the algebra.
2. The structure matrix of $\mathscr{L}$ (with product $*$) is defined as

$$M_{\mathscr{L}} = \begin{bmatrix} \alpha_{11}^1 & \cdots & \alpha_{1n}^1 & \cdots & \alpha_{n1}^1 & \cdots & \alpha_{nn}^1 \\ \alpha_{11}^2 & \cdots & \alpha_{1n}^2 & \cdots & \alpha_{n1}^2 & \cdots & \alpha_{nn}^2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{11}^n & \cdots & \alpha_{1n}^n & \cdots & \alpha_{n1}^n & \cdots & \alpha_{nn}^n \end{bmatrix}. \tag{16.23}$$

Fix this basis and let vectors $X=\sum_{i=1}^n x_i e_i$, $Y=\sum_{i=1}^n y_i e_i$ etc. be expressed in vector form as $X=(x_1,\cdots,x_n)^{\mathrm{T}}$, $Y=(y_1,\cdots,y_n)^{\mathrm{T}}$ etc. Then the following proposition is easily verifiable.

Proposition 16.2. *Let $Z=X*Y$. Then the vector form of Z can be calculated as*

$$Z = M_{\mathscr{L}} \ltimes X \ltimes Y = M_{\mathscr{L}} XY. \tag{16.24}$$

It is clear that all the properties of an algebra are determined by it structure matrix. In the following we investigate some basic properties of an algebra via its structure matrix. For notational ease, the symbol $\ltimes$ is omitted.

Definition 16.5. Given an algebra $\mathscr{L}$.

1. $\mathscr{L}$ is said to be symmetric, if

$$X*Y = Y*X, \quad \forall X,Y \in \mathscr{L}; \tag{16.25}$$

2. $\mathscr{L}$ is said to be skew-symmetric, if

$$X*Y = -Y*X, \quad \forall X,Y \in \mathscr{L}; \tag{16.26}$$

3. $\mathscr{L}$ satisfying associative rule, if

$$(X*Y)*Z = X*(Y*Z), \quad \forall X,Y,Z \in \mathscr{L}. \tag{16.27}$$

Proposition 16.3. *Given an n-dimensional algebra $\mathscr{L}$.*

1. $\mathscr{L}$ is symmetric, if and only if

$$M_{\mathscr{L}}(W_{[n]} - I_{n^2}) = 0; \tag{16.28}$$

2. $\mathscr{L}$ is skew symmetric, if and only if

$$M_{\mathscr{L}}(W_{[n]}+I_{n^2})=0; \tag{16.29}$$

3. $\mathscr{L}$ satisfies associative rule, if and only if

$$M_{\mathscr{L}}(M_{\mathscr{L}}\otimes I_n - I_n\otimes M_{\mathscr{L}})=0. \tag{16.30}$$

To prove this proposition we need a lemma, which itself is useful.

Recalling Definition 2.6 and Proposition 2.13, we know that let V_1 and V_2 be two vector spaces with dimensions n_1 and n_2 respectively. The tensor product space $V=V_1\otimes V_2$ is generated by $\{x\otimes y\,|\,x\in V_1, y\in V_2\}$. Furthermore, assume $\Phi: V_1\times V_2\to W$ is a bilinear mapping, then there exists a unique mapping $\Psi: V\to W$, which makes the graph 16.1 commutative.

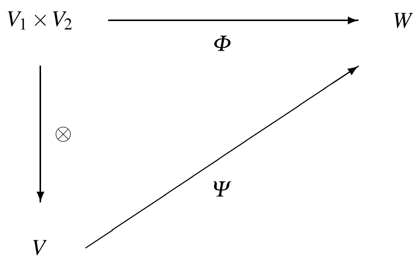

Fig. 16.1 Tensor Product Space

This concept can be extended to the tensor product of k vector spaces deduced by k-th fold linear mapping.

Let V_i, $i = 1, \cdots, k$ be n_i-dimensional vector spaces with their bases $\{d^i_1, \cdots, d^i_{n_i}\}$ and W an m-dimensional vector space with a fixed basis. Corresponding to these basis a multi-linear mapping $\Phi : V_1 \times \cdots \times V_k \to W$ has its structure matrix $M_\Phi \in \mathscr{M}_{m\times n}$, where $n = \prod_{i=1}^k n_i$. Tensor product space $V = T_{i=1}^k V_i$ has basis

$$\begin{aligned}\{e_1, \cdots, e_n\} := \Big\{ e^1_{i_1} \otimes e^2_{i_1} \otimes \cdots \otimes e^k_{i_k} \,\Big|\, & i_k = 1, \cdots, n_k, \\ & i_{k-1} = 1, \cdots, n_{k-1}, \cdots, i_1 = 1, \cdots, n_1 \Big\}. \end{aligned} \tag{16.31}$$

Then there exists a unique deduced mapping $\Psi : V \to W$ such that Φ and $\Psi \circ \otimes$ commutative.

Using these concepts, we can prove the following lemma.

Lemma 16.2. *1. Under the basis defined in (16.31), Ψ and Φ have the same structure matrices, i.e.,*

$$M_\Psi = M_\phi. \tag{16.32}$$

2. If

$$\Phi(X_1, \cdots, X_k) = 0, \quad \forall X_i \in V_i,$$

then

$$\Psi(Y) = 0, \quad \forall Y \in V.$$

Proof. 1. From the construction of the mapping one sees that

$$\Psi(e^1_{i_1} \otimes \cdots \otimes e^k_{i_k}) = \Phi(e^1_{i_1}, \cdots, e^k_{i_k}), \quad \forall 1 \le i_t \le n_t,\ t = 1, \cdots, k.$$

Note that

$$\Big\{ e^1_{i_1} \otimes \cdots \otimes e^k_{i_k} \,\Big|\, 1 \le i_t \le n_t, t = 1, \cdots, k \Big\}$$

form a basis of V. That is, (16.32) holds on the basis of V. Since $\Psi : V \to W$ is a linear mapping, (16.32) holds for overall V.

2. It is a direct consequence of (1), In fact, it comes from (1) by setting $M_\phi = 0$. □

Proof (Proof of Proposition 16.3). We prove (16.30) only. The proof of (16.28) or (16.29) is similar.

Using (16.24), equation (16.27) can be expressed in matrix form as

$$MM(XY)Z = MX(MYZ), \quad \forall X,Y,Z \in \mathscr{L}.$$

Using associativity, we have

$$M^2XYZ = M(XM)YZ = M(I_n \otimes M)XYZ, \quad \forall X,Y,Z \in \mathscr{L}.$$

Then we have

$$(M^2 - M(I_n \otimes M))XYZ = 0, \quad \forall X,Y,Z \in \mathscr{L}. \tag{16.33}$$

Note that though

$$S = \{XYZ \,|\, X,Y,Z \in \mathscr{L}\}$$

is not a vector space, Lemma 16.2 assures the correctness of (16.33) for any X, Y, Z in the vector space. (16.30) follows. □

Next, we consider when an algebra is a Lie algebra.

Proposition 16.4. *An algebra $\mathscr{L}$ is a Lie algebra, if and only if its structure matrix satisfies*

(i) sky-symmetry: That is, (16.29) holds;
(ii) Jacobi identity: That is,

$$M^2(I_{n^2} + W_{[n,n^2]} + W_{[n^2,n]}) = 0. \tag{16.34}$$

Proof. It is well know that an algebra is a Lie algebra, if and only if it is skew symmetry and satisfies Jacobi identity:

$$(X * Y) * Z + (Y * Z) * X + (Z * X) * Y = 0, \quad \forall X,Y,Z \in \mathscr{L}. \tag{16.35}$$

According to Proposition 16.3, skew symmetry is equivalent to (16.29), which leads to (i).

As for (ii), using structure matrix, (16.35) can be expressed as

$$M^2(XYZ + YZX + ZXY) = 0. \tag{16.36}$$

Using the proposition of swap matrix, we have

$$W_{[n,n^2]}XYZ = YZX, \quad W_{[n^2,n]}XYZ = ZXY.$$

Plugging it into (16.36), an argument similar to the proof of Proposition 16.3 shows (16.35). □

Example 16.1. In $\mathbb{R}^3$ the cross product is defined as follows: Let $X = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $Y = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$. Then

$$X \times Y = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Its structure matrix is easily calculated as

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{16.37}$$

A straightforward computation shows that

$$M\left(I_9 + W_{[3]}\right) = 0,$$

and

$$M^2\left(I_{27} + W_{[3.9]} + W_{[9,3]}\right) = 0.$$

Hence $\mathbb{R}^3$ wit cross product is a Lie algebra.

Next, we consider the structure matrix of general linear algebra $gl(n,\mathbb{R})$. First, we give a formula for the column stacking form of the product of two matrices, which itself is useful.

Lemma 16.3. *Let $A \in \mathscr{M}_{m\times n}$ and $B \in \mathscr{M}_{n\times p}$. Then*

$$V_c(AB) = \Psi^n_{mp} V_c(A) V_c(B), \tag{16.38}$$

where

$$\Psi^n_{mp} = \begin{bmatrix} I_m \otimes (\delta^1_p \delta^1_n)^{\mathrm{T}} & I_m \otimes (\delta^1_p \delta^2_n)^{\mathrm{T}} & \cdots & I_m \otimes (\delta^1_p \delta^n_n)^{\mathrm{T}} \\ I_m \otimes (\delta^2_p \delta^1_n)^{\mathrm{T}} & I_m \otimes (\delta^2_p \delta^2_n)^{\mathrm{T}} & \cdots & I_m \otimes (\delta^2_p \delta^n_n)^{\mathrm{T}} \\ \vdots & \vdots & & \vdots \\ I_m \otimes (\delta^p_p \delta^1_n)^{\mathrm{T}} & I_m \otimes (\delta^p_p \delta^2_n)^{\mathrm{T}} & \cdots & I_m \otimes (\delta^p_p \otimes \delta^n_n)^{\mathrm{T}} \end{bmatrix}, \tag{16.39}$$

where $\delta^i_p \mathrm{Col}_i(I_p)$ ($\delta^n_i = \mathrm{Col}_i(I_n)$).

Observing Table **??**, it is easy to verify that

$$V_c(AB) = (I_p \otimes A) V_c(B).$$

Hence we have to calculate $I_p \otimes A$. An easy calculation shows

$$I_p \otimes A = \Psi_{mnp} V_c(A).$$

Now we are ready to consider the structure matrix of $gl(n,\mathbb{R})$. For notational compactness, we denote by $\Psi_n := \Psi^n_{nn}$.

Example 16.2. Consider $gl(n,\mathbb{R})$. Choose a basis as $\{M_{I,J},\ I=1,\cdots,n;\ J=1,\cdots,n\}$, where $M_{I,J}$ is defined by

$$(M_{I,J})_{i,j}=\begin{cases}1, & i=I \text{ and } j=J,\\ 0, & \text{Otherwise.}\end{cases}$$

Define the product, called the Lie bracket, as

$$[A,B]=AB-BA.$$

We construct its structure matrix. Using Lemma 16.3, we have

$$V_c(AB)=\Psi_n V_c(A)V_c(B),$$

and

$$V_c(BA)=\Psi_n V_c(B)V_c(A)=\Psi_n W_{[n^2]}V_c(A)V_c(B).$$

Hence,

$$V_c([A,B])=(\Psi_n-\Psi_n W_{[n^2]})V_c(A)V_c(B),$$

We conclude that the structure matrix of $gl(n,\mathbb{R})$ is

$$M=\Psi_n(I_{n^4}-W_{[n^2]}). \tag{16.40}$$

We leave to the reader to verify that M satisfies (16.29) and (16.34).

In the following we consider the topological structure of the n-dimensional real algebra. Note that an algebra is uniquely determined by its structure matrix, we there can pose the conventional topological structure of $\mathbb{R}^{n\times n^2}$ to n-dimensional algebra. Since the structure matrix depends on the coordinate frame. The different structure matrices caused by coordinate transformations should be equivalent. The quotient topology under this equivalence then becomes a proper description of the algebra space.

Let $M_{\mathscr{L}_n}$ be the structure matrix of $\mathscr{L}_n$ with respect to the basis $\{e_1\}$. Let e_2 be another basis of $\mathbb{R}^n$ and there exists a linear transformation $T\in GL(n,\mathbb{R})$, such that $e_2=e_1T$. Here $GL(n,\mathbb{R})$ is the n-dimensional general linear group. Now, consider two vectors $V_1=e_1X_1$ and $V_2=e_1X_2$. Using new basis, their vector forms become $\tilde{X}_i=T^{-1}X_i$, $i=1,2$. Then we have

$$\begin{aligned}T^{-1}M_{\mathscr{L}_n}X_1X_2&=\tilde{M}_{\mathscr{L}_n}T\tilde{X}_1T\tilde{X}_2\\ &=\tilde{M}_{\mathscr{L}_n}T^{-1}(I_n\otimes T^{-1})X_1X_2=\tilde{M}_{\mathscr{L}_n}(T^{-1}\otimes T^{-1})X_1X_2.\end{aligned}$$

Therefore, we have

$$\tilde{M}_{\mathscr{L}_n}=T^{-1}M_{\mathscr{L}_n}(T\otimes T). \tag{16.41}$$

Using this result to two algebras, we have

Proposition 16.5. *Consider two n-dimensional algebras $\mathscr{L}_n^1$ and $\mathscr{L}_n^2$. They are equivalent, if and only if there exists a $T \in GL(n,\mathbb{R})$ such that*

$$M_{\mathscr{L}_n^1} = T^{-1} M_{\mathscr{L}_n^2} (T \otimes T). \tag{16.42}$$

Hence the topology of n-dimensional algebras is

$$\mathbb{R}^{n \times n^2} / GL(n,\mathbb{R}).$$

Another interesting topic is the structure of Lie algebra. First, we consider the case of $n = 2$. To assure the skew symmetry, a 2-dimensional Lie algebra has its structure matrix as

$$M_{\mathscr{L}_2} = \begin{bmatrix} 0 & a & -a & 0 \\ 0 & b & -b & 0 \end{bmatrix}. \tag{16.43}$$

A simple computation shows that

$$M_{\mathscr{L}_2}^2 = \begin{bmatrix} 0 & 0 & -ab & a^2 & ab & -a^2 & 0 & 0 \\ 0 & 0 & -b^2 & ab & b^2 & -ab & 0 & 0 \end{bmatrix}. \tag{16.44}$$

We also have

$$I_8 + W_{[2,4]} + W_{[4,2]} = \begin{bmatrix} 3&0&0&0&0&0&0&0 \\ 0&1&1&0&1&0&0&0 \\ 0&1&1&0&1&0&0&0 \\ 0&0&0&1&0&1&1&0 \\ 0&1&1&0&1&0&0&0 \\ 0&0&0&1&0&1&1&0 \\ 0&0&0&1&0&1&1&0 \\ 0&0&0&0&0&0&0&3 \end{bmatrix}. \tag{16.45}$$

It is ready to check that

$$M_{\mathscr{L}_2}^2 (I_8 + W_{[2,4]} + W_{[4,2]}) = 0.$$

Hence we have the following result:

Proposition 16.6. *Any 2-dimensional algebra is Lie algebra.*

Next, we consider the case of $n = 3$. To assure skew symmetry, its structure matrix must be as

$$M_{\mathscr{L}_3} = \begin{bmatrix} 0 & a & d & -a & 0 & g & -d & -g & 0 \\ 0 & b & e & -b & 0 & h & -e & -h & 0 \\ 0 & c & f & -c & 0 & i & -f & -i & 0 \end{bmatrix}. \tag{16.46}$$

A simple computer routine can calculate

$$M_{\mathscr{L}_3}^2 (I_{27} + W_{[3,9]} + W_{[9,3]}),$$

which is a 3×27 matrix. Fortunately, it has only a few non-zero elements, which are

$$\begin{aligned}
m_{1,6} = m_{1,16} = m_{1,22} = -m_{1,8} = -m_{1,12} = -m_{1,20} = bg + gf - ah - di;\\
m_{2,6} = m_{2,16} = m_{2,22} = -m_{2,8} = -m_{2,12} = -m_{2,20} = ae - bd + hf - ei;\\
m_{3,6} = m_{3,16} = m_{3,22} = -m_{3,8} = -m_{3,12} = -m_{3,20} = af + bi - cd - ch.
\end{aligned}$$

We conclude that

Theorem 16.3. *A 3-dimensional algebra is a Lie algebra, if and only if its structure matrix has the form of (16.46), where the non-zero elements satisfy the following equations:*

$$\begin{cases} bg + gf - ah - di = 0,\\ ae - bd + hf - ei = 0,\\ af + bi - cd - ch = 0. \end{cases} \tag{16.47}$$

Example 16.3. According to Theorem 16.3, we are able to construct many 3-dimensional algebras.

1. Set
$$a = b = d = f = h = i = 0, \quad c = g = 1, \quad e = -1.$$
It is easy to check that this is a solution of (16.47). In fact, this solution corresponds to the standard cross product on $\mathbb{R}^3$, which is a known Lie algebra.
2. To get another non-trivial solution of (16.47), we convert it into a matrix form.
$$\begin{bmatrix} -h & g & 0\\ e & -d & 0\\ f & i & -d-h \end{bmatrix} \begin{bmatrix} a\\ b\\ c \end{bmatrix} = \begin{bmatrix} di - gf\\ ei - hf\\ 0 \end{bmatrix}. \tag{16.48}$$
To find a non-zero solution of (16.47) it suffices to choose d, e, f, g, h, i, such that the coefficient matrix of (16.48) is non-singular. Then we can uniquely solve the corresponding a, b, c. For instance, we choose $d = -e = 1$, $f = -g = 2$, $h = -i = 3$, Then we have
$$\begin{bmatrix} -3 & -2 & 0\\ -1 & -1 & 0\\ 2 & -3 & -4 \end{bmatrix} \begin{bmatrix} a\\ b\\ c \end{bmatrix} = \begin{bmatrix} 1\\ -3\\ 0 \end{bmatrix}.$$
Its solution is $a = -7$, $b = 10$, $c = -11$. Hence, we can have another Lie algebra as
$$\mathscr{L}_3 = \{\alpha I + \beta J + \gamma K \,|\, \alpha, \beta, \gamma \in \mathbb{R}\},$$
with its product $*$ defined as

$$\begin{aligned}
&I*I=J*J=K*K=0\\
&I*J=-J*I=-7I+10J-11K\\
&I*K=-K*I=I-J+2K\\
&J*K=-K*J=-2I+3J-3K.
\end{aligned}$$

We continue to consider the case of $n=4$. To assure skew symmetry its 4×16 structure matrix, denoted by $M_{\mathscr{L}_4}=(m_{ij})$, should satisfy

$$\begin{cases}
m_{i,j}=0, \quad j=1,6,11,16;\\
m_{i,2}=-m_{i,5}:=x_i,\\
m_{i,3}=-m_{i,9}:=x_{4+i},\\
m_{i,4}=-m_{i,13}:=x_{8+i},\\
m_{i,7}=-m_{i,10}:=x_{12+i},\\
m_{i,8}=-m_{i,14}:=x_{16+i},\\
m_{i,12}=-m_{i,15}:=x_{20+i},\\
\qquad\qquad\qquad i=1,2,3,4.
\end{cases} \tag{16.49}$$

Using Matlab, a simple routine converts (16.34) to its equivalent form as

$$\begin{cases}
-x_1x_{14}+x_2x_{13}-x_4x_{21}-x_5x_{15}+x_7x_{13}+x_8x_{17}-x_9x_{16}=0,\\
x_1x_6-x_2x_5-x_4x_{22}-x_6x_{15}+x_7x_{14}+x_8x_{18}-x_{10}x_{16}=0,\\
x_1x_7+x_2x_{15}-x_3x_5-x_3x_{14}-x_4x_{23}+x_8x_{19}-x_{11}x_{16}=0,\\
x_1x_8+x_2x_{16}-x_4x_5-x_4x_{14}-x_4x_{24}+x_7x_{16}-x_8x_{15}\\
\quad +x_8x_{20}-x_{12}x_{16}=0,\\
-x_1x_{18}+x_2x_{17}+x_3x_{21}-x_5x_{19}-x_9x_{20}+x_{11}x_{13}+x_{12}x_{17}=0,\\
x_1x_{10}-x_2x_9+x_3x_{22}-x_6x_{19}-x_{10}x_{20}+x_{11}x_{14}+x_{12}x_{18}=0,\\
x_1x_{11}+x_2x_{19}-x_3x_9-x_3x_{18}+x_3x_{23}-x_7x_{19}+x_{11}x_{15}\\
\quad -x_{11}x_{20}+x_{12}x_{19}=0,\\
x_1x_{12}+x_2x_{20}+x_3x_{24}-x_4x_9-x_4x_{18}-x_8x_{19}+x_{11}x_{16}=0,\\
-x_1x_{22}-x_5x_{23}+x_6x_{17}+x_7x_{21}-x_9x_{24}-x_{10}x_{13}+x_{12}x_{21}=0,\\
-x_2x_{22}+x_5x_{10}-x_6x_9+x_6x_{18}-x_6x_{23}+x_7x_{22}-x_{10}x_{14}\\
\quad -x_{10}x_{24}+x_{12}x_{22}=0,\\
-x_3x_{22}+x_5x_{11}+x_6x_{19}-x_7x_9-x_{10}x_{15}-x_{11}x_{24}+x_{12}x_{23}=0,\\
-x_4x_{22}+x_5x_{12}+x_6x_{20}+x_7x_{24}-x_8x_9-x_8x_{23}-x_{10}x_{16}=0,\\
x_1x_{21}-x_5x_{17}+x_9x_{13}-x_{13}x_{18}-x_{13}x_{23}+x_{14}x_{17}\\
\quad +x_{15}x_{21}-x_{17}x_{24}+x_{20}x_{21}=0,\\
x_2x_{21}-x_6x_{17}+x_{10}x_{13}-x_{14}x_{23}+x_{15}x_{22}-x18x_{24}+x_{20}x_{22}=0,\\
x_3x_{21}-x_7x_{17}+x_{11}x_{13}+x_{14}x_{19}-x_{15}x_{18}-x_{19}x_{24}+x_{20}x_{23}=0,\\
x_4x_{21}-x_8x_{17}+x_{12}x_{13}+x_{14}x_{20}+x_{15}x_{24}-x_{16}x_{18}\\
\quad -x_{16}x_{23}=0.
\end{cases} \tag{16.50}$$

Then we have

Theorem 16.4. *A 4-dimensional algebra is a Lie algebra, if and only if its structure matrix satisfies (16.49) with its non-zero parameters $x_1,\cdots,x_{24}$ satisfy (16.50).*

Example 16.4. 1. Consider $gl(2,\mathbb{R})$. Similar to 16.2, we choose its basis as $\{e_1,e_2,e_3,e_4\}$, where

$$e_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad e_2=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad e_3=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad e_4=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

A straightforward computation shows that its structure matrix is

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. In fact, (16.50) has many solutions. For instance, choosing

$$x_i = 0, \quad i > 8,$$

then (16.50) becomes

$$\begin{cases} x_1x_6 - x_2x_5 = 0, \\ x_1x_7 - x_3x_5 = 0, \\ x_1x_8 - x_4x_5 = 0. \end{cases} \tag{16.51}$$

It is obvious that $\{x_1, x_2, \cdots, x_8\}$ is a solution of (16.50), if and only if

$$x_1 : x_2 : x_3 : x_4 = x_5 : x_6 : x_7 : x_8.$$

For instance, a simple solution is $x_1 = 1$, $x_2 = -1$, $x_3 = 2$, $x_4 = -2$, $x_5 = -1$, $x_6 = 1$, $x_7 = -2$, $x_8 = 2$. Using it, we have a Lie algebra as

$$\mathscr{L} = \{aI + bJ + cK + dH \,|\, a,b,c,d \in \mathbb{R}\},$$

with its product $*$ satisfying

$$\begin{cases} I*I = J*J = K*K + H*H = 0, \\ I*J = -J*I = -I + J - 2K + 2H, \\ I*K = -K*J = I - J + 2K - 2H, \\ I*H = -H*I = J*K = -K*J = J*H \\ \quad = -H*J = K*H = -H*K = 0. \end{cases}$$

Finally, w consider the general case where the dimension is n. Let the structure matrix of an n-dimensional algebra be expressed as

$$M_{\mathscr{L}_n} = \begin{bmatrix} W_{11} \cdots W_{1n} & \cdots & W_{n1} \cdots W_{nn} \end{bmatrix},$$

where each W_{ij} is a column with w_{ij}^s as its elements. Then we can calculate that

$$M_{\mathscr{L}_n^2} = \left[\sum_{s=1}^{n} w_{ij}^s W_{sk} \,\middle|\, i,j,k = 1, \cdots, n \right]. \tag{16.52}$$

Note that the columns of the matrix in (16.52) are arranged by the order of $\mathrm{id}(i,j,k;n,n,n)$. When right multiply it by $W_{[n^2,n]}$ (or $W_{[n,n^2]}$), its order becomes id$(j,k,i;n,n,n)$ (correspondingly, id$(k,i,j;n,n,n)$). Taking this into consideration, we have

$$\sum_{s=1}^{n} w_{ij}^{s} W_{sk} + \sum_{s=1}^{n} w_{jk}^{s} W_{si} + \sum_{s=1}^{n} w_{ki}^{s} W_{sj} = 0.$$

Because of the skew symmetry we need only to consider the case when i, j, k are distinct to each other. Hence, we have

$$\begin{aligned} &\sum_{s<k} w_{ij}^{s} W_{sk} - \sum_{s>k} w_{ij}^{s} W_{ks} + \sum_{s<i} w_{jk}^{s} W_{si} - \sum_{s>i} w_{jk}^{s} W_{is} + \sum_{s<j} w_{ki}^{s} W_{sj} \\ &\quad - \sum_{s>j} w_{ki}^{s} W_{js} = 0, \quad 1 \le i < j < k \le n. \end{aligned} \tag{16.53}$$

Hence we have

$$\frac{k!n}{3!(k-3)!}$$

independent equations. The set of Lie algebra is the algebraic variety generated by this set of polynomial equations [5].

Another interesting topic is the invertibility of algebra.

Definition 16.6. Given an algebra $\mathscr{L}$ with its multiplication $*$. $\mathscr{L}$ is said to be invertible if

(i) there exists a unitary element e, such that

$$e * x = x, \quad \text{and} \quad x * e = x, \quad \forall x \in \mathscr{L}; \tag{16.54}$$

(ii) for any element $x \neq 0$, there exists unique element, denoted by $x^{-1} \in \mathscr{L}$, such that

$$x * x^{-1} = e. \tag{16.55}$$

If in addition, if

(iii) $\mathscr{L}$ is commutative, that is,

$$x * y = y * x, \quad \forall x, y \in \mathscr{L}, \tag{16.56}$$

then $\mathscr{L}$ is called a field.

Note that since $0 \neq e \in \mathscr{L}$, it can be considered as a basis element. In fact, the one-dimensional subspace generated by e is isomorphic to $\mathbb{R}$. For convenience, if there is an unitary e, we always choose e as the first element of a basis of $\mathscr{L}$. That is, a conventional basis is $B = \{e_1 = e, e_2, e_3, \cdots, e_n\}$. In vector form we have $e = (1, 0, \cdots, 0)^{\mathrm{T}}$.

Assume the basis B is fixed, with respect to B we have the structure matrix M of $\mathscr{L}$. Then we consider how to check the condition (i) and (ii) of Definition 16.6. Split M as

$$M = \left[M_1 \; M_2 \; \cdots \; M_n\right],$$

where $M_i = \mathrm{Blk}_i(M)$ are $n \times n$ matrices. The following result is an immediate consequence of the definition. First, we consider condition (i):

Lemma 16.4. *There exists the unitary* $e = (1, 0, \cdots, 0)^{\mathrm{T}}$*, if and only if* $M_1 = I_n$*,* $\mathrm{Col}_1(M_j) = \delta_n^j$.

Proof.

$$e * x = Mex = [M_1, \cdots, M_n]\, \delta_1^n x = M_1 x.$$

Then $e * x = x$ leads to

$$M_1 x = x, \quad \forall x \in \mathbb{R}^n.$$

It follows that $M_1 = I_n$.

Similarly, since

$$x * e = MW_{[n]} ex = \left[M_1^1, \cdots, M_n^1\right] x = x, \quad \forall x \in \mathbb{R}^n,$$

which means $\left[M_1^1, \cdots, M_n^1\right] = I_n$. □

Next, we consider consider condition (ii). What we need to verify is: for any $x \neq 0$, there exists a unique y, such that

$$Mx * y = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{16.57}$$

Then, it is not to verify that

Lemma 16.5. *Assume* $\mathscr{L}$ *has unitary* $e = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}$*, then for any* $x \neq 0$ *there is a unique inverse element* x^{-1}*, if and only if*

$$\det(Mx) \neq 0, \quad \forall x \neq 0. \tag{16.58}$$

We investigate some examples.

Example 16.5. Consider the set of complex numbers $\mathbb{C}$, which can be considered as a vector space over $\mathbb{R}$, which has basis $\{1, i\}$. Then its structure matrix is

$$M_{\mathbb{C}} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

We verify the commutativity, condition (i), and condition (ii).

1. Commutativity: Note that

$$W_{[2]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that

$$M_{\mathbb{C}}W_{[2]} = M_{\mathbb{C}}.$$

2. Condition (i): Let $x = (\alpha,\beta)^{\mathrm{T}} \sim \alpha + \beta i \in \mathbb{C}$. Then

$$\begin{aligned} x * e = M_{\mathbb{C}}xe &= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \end{aligned}$$

3. Condition (ii):

$$\det(M_{\mathbb{C}}x) = \det\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = \alpha^2 + \beta^2.$$

It follows that $\det(M_{\mathbb{C}}x) = 0$, if and only if $x = 0$.

Example 16.6. Consider the quaternion [4], its standard basis is $\{1, I, J, K\}$. Under this basis its structure matrix is

$$M_Q = \begin{bmatrix} 1&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&-1 \\ 0&1&0&0&1&0&0&0&0&0&0&1&0&0&-1&0 \\ 0&0&1&0&0&0&0&-1&1&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&1&0&0&-1&0&0&1&0&0&0 \end{bmatrix}. \tag{16.59}$$

It is easy to verify condition (i). As for condition (ii), assume $x = \begin{bmatrix} a & b & c & d \end{bmatrix}^{\mathrm{T}} \neq 0$, we have

$$M_Q x = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix},$$

and

$$\begin{aligned} E :&= \det(M_Q x) \\ &= a^4 + b^4 + c^4 + d^4 + 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) \\ &= (a^2 + b^2 + c^2 + d^2)^2 > 0. \end{aligned} \tag{16.60}$$

Hence the quaternion is invertible. Moreover, we have

$$x^{-1} = (M_Q X)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} := \frac{1}{E} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix},$$

where

$$\alpha = \det\left(\begin{bmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{bmatrix}\right) = a^3 + a(b^2 + c^2 + d^2);$$

$$\beta = -\det\left(\begin{bmatrix} b & -d & c \\ c & a & -b \\ d & b & a \end{bmatrix}\right) = -b^3 - b(a^2+c^2+d^2);$$

$$\gamma = \det\left(\begin{bmatrix} b & a & c \\ c & d & -b \\ d & -c & a \end{bmatrix}\right) = -c^3 - c(a^2+b^2+d^2);$$

$$\delta = -\det\left(\begin{bmatrix} b & a & -d \\ c & d & a \\ d & -c & b \end{bmatrix}\right) = -d^3 - d(a^2+b^2+c^2);$$

It is not a field, because it is not commutative.

Remark 16.2. Note that if $\mathscr{L}$ is commutative, then $x*x^{-1} = e$ implies $x^{-1}*x = e$. In general, Definition 16.6 defines only the right inverse. If the right (left) inverse exists, the commutativity assures the exists of left (right) inverse, and the equivalence of the two inverses.

In the following we give a weak condition for the existence and equivalence of the left and right inverse.

Proposition 16.7. *Assume $\mathscr{L}$ is an associative algebra with right (left) unitary, and its non-zero elements have right (left) inverse. Then the right (left) inverse of each non-zero element is also its left (right) inverse.*

Proof. It is obvious that all the non-zero elements with the algebra product form a group. Then the conclusion comes from the corresponding property of groups. □

The above property motivates a rational question: when an algebra is associative? We have the following result.

Proposition 16.8. *An n-dimensional algebra $\mathscr{L}$ is associative, if and only if its structure matrix satisfies the following condition.*

$$M_{\mathscr{L}}^2 = M_{\mathscr{L}}(I_n \otimes M_{\mathscr{L}}). \tag{16.61}$$

Proof.

$$\begin{aligned} (x*y)*z &= M_{\mathscr{L}}(M_{\mathscr{L}}xy)z = M_{\mathscr{L}}^2 xyz, \\ x*(y*z) &= M_{\mathscr{L}}x(M_{\mathscr{L}}yz) = M_{\mathscr{L}}xM_{\mathscr{L}}yz = M_{\mathscr{L}}(I_n \otimes M_{\mathscr{L}})xyz. \end{aligned}$$

The conclusion follows. □

Example 16.7. Consider the quaternion in Example 16.6. It is easy to verify that its structure matrix M_Q satisfies (16.61). Hence the left inverse of the quaternion is also the right inverse.

In the following we consider whether we can find a new algebra over $\mathbb{R}$, which is a field.

First, we consider the case when the dimension $n = 2$. Assume $\{1, \xi\}$ is a basis, which makes

$$\mathscr{F}_2 := \{a + b\xi \mid a, b \in \mathbb{R}\}$$

a field. We need to define a product. Since it needs to satisfy the commutativity requirement and the condition (i), its structure matrix must have the form as

$$M = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 1 & \beta \end{bmatrix}. \tag{16.62}$$

Now we consider the condition (ii). Then for any $x = (a, b)^{\mathrm{T}}$ we have

$$\det(Mx) = a^2 + \beta ab - \alpha b^2.$$

To assure $\det(Mx) > 0$, $\forall x \neq 0$, we need

$$\Delta = \beta^2 + 4\alpha < 0. \tag{16.63}$$

We then have the following result.

Theorem 16.5. *A 2-dimensional algebra over $\mathbb{R}$, which is a field, if and only if for a standard basis $(1, \xi)$, its structure matrix has the form of (16.62), where $\alpha, \beta \in \mathbb{R}$ satisfy*

$$|\beta| < 2\sqrt{-\alpha}.$$

Remark 16.3. The following two facts come from equation (16.62):

1.

$$\xi^2 = \alpha + \beta\xi. \tag{16.64}$$

2. If only $x = (a, b)^{\mathrm{T}} \neq 0$, its inverse is

$$(a + b\xi)^{-1} = \frac{1}{a^2 + \beta ab - \alpha b^2}\left[(a + \beta b) - b\xi\right]. \tag{16.65}$$

Example 16.8. Define a 2-dimensional algebra $\mathscr{J}$ as

$$\mathscr{J} = \{a + b\mathbf{j} \mid \alpha, \beta \in \mathbb{R}\},$$

with its structure matrix as

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{16.66}$$

According to Theorem 16.5, it is a field. Using (16.64), we have

$$\mathbf{j}^2 = -1 + \mathbf{j}. \tag{16.67}$$

Now the product of two numbers in $\mathscr{J}$ can be calculated via (16.67). For instance,

$$(3+2\mathbf{j})(2-\mathbf{j}) = 6+\mathbf{j}-2\mathbf{j}^2 = 8-\mathbf{j}.$$

The division can be performed by using (16.65). For instance, consider $(3+2\mathbf{j})/(2-\mathbf{j})$, using (16.65), we have

$$\frac{1}{2-\mathbf{j}} = \frac{1}{3}(1+\mathbf{j}).$$

Hence,

$$\frac{3+2\mathbf{j}}{2-\mathbf{j}} = \frac{1}{3}+\frac{7}{3}\mathbf{j}.$$

In fact, we did not find new field. See that following proposition.

Proposition 16.9. *Any two 2-dimensional fields are isomorphic.*

Proof. Let $\mathscr{J} = \{a+b\mathbf{j}\,|\,a,b\in\mathbb{R}\}$ be a two-dimensional field with its structure matrix as

$$M_{\mathscr{J}} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 1 & \beta \end{bmatrix}.$$

Define a linear mapping $\Phi : \mathscr{J} \to \mathbb{C}$ as

$$1 \mapsto 1, \quad \mathbf{j} \mapsto \frac{\beta}{2}+\frac{\mathbf{i}}{2}\sqrt{-(4\alpha+\beta^2)}.$$

It is easy to verify that Φ is an isomorphism. Hence, any two 2-dimensional field over $\mathbb{R}$ is isomorphic to the field of complexity $\mathbb{C}$. □

Example 16.9. Given a 3-dimensional algebra $\mathscr{L}_3$ over $\mathbb{R}$. Assume it is a field, then for a standard basis $\{e,I,J\}$, where e is the unitary. Assume it is symmetric, then its structure matrix should have the following form.

$$M_{\mathscr{L}_3} = \begin{bmatrix} 1 & 0 & 0 & 0 & a & d & 0 & d & g \\ 0 & 1 & 0 & 1 & b & e & 0 & e & h \\ 0 & 0 & 1 & 0 & c & f & 1 & f & i \end{bmatrix}. \tag{16.68}$$

Now we check when the condition (ii) is satisfied. Assume $w=(x,y,z)^{\mathrm{T}}\in\mathscr{V}$, then

$$\det\left(M_{\mathscr{L}_3}w\right) = \det\left(\begin{bmatrix} x & ay+dz & dy+gz \\ y & x+by+ez & ey+hz \\ z & cy+fz & z+fy+iz \end{bmatrix}\right) = x^3+LDT(x),$$

where $LDT(x)$ express the lower order terms of x. Now it is clear that it can not be positive definite. We conclude that there is no three-dimensional field.

In fact, Weierstrass proved in 1861 that finite dimensional algebra over $\mathbb{R}$, which satisfies associativity and commutativity, can only be either the field of real numbers $\mathbb{R}$ or field of complex numbers $\mathbb{C}$.

In the following, we give an almost invertible 4-dimensional commutative algebra with identity.

Example 16.10. Let $\mathscr{L}$ be a 4-dimensional commutative algebra with identity. Moreover, its structure matrix has the following form

$$M_{\mathscr{L}} = \begin{bmatrix} 1&0&0&0&0&-1&0&0&0&0&1&0&0&0&0&-1 \\ 0&1&0&0&1&0&0&0&0&0&0&-1&0&0&-1&0 \\ 0&0&1&0&0&0&0&1&1&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&-1&0&0&-1&0&0&1&0&0&0 \end{bmatrix}. \tag{16.69}$$

Let $\xi = (x,y,z,w)^{\mathrm{T}} \in \mathbb{R}^4$. A straightforward computation shows that

$$\det(M_{\mathscr{L}}\xi) = (x^2 - z^2)^2 + (y^2 - w^2)^2 + 2(xy + zw)^2 + 2(xw + yz)^2.$$

Hence, $x + iy + jz + kw$ is invertible, if and only if

$$\begin{bmatrix} x \\ y \end{bmatrix} \neq \pm \begin{bmatrix} z \\ w \end{bmatrix}.$$

$\mathscr{L}$ is invertible except a zero-measure set.

$$\left\{ (x,y,z,w)^{\mathrm{T}} \in \mathbb{R}^4 \,\middle|\, (x,y) = \pm(z,-w) \right\}.$$

Finally, we consider in general how to calculate $\det(M\xi)$ (as required in (16.58))
.

Assume an n-dimensional algebra has its structure matrix as

$$M_n = \begin{bmatrix} W_{11} \cdots W_{1n} & \cdots & W_{n1} \cdots W_{nn} \end{bmatrix}, \tag{16.70}$$

where $W_{ij} \in \mathbb{R}^n$ are the columns of M_n arranged in the order of $\mathrm{id}(i,j;n,n)$. We deduce the formula for $\det(M_n\xi)$.

Denote by $k_1, \cdots, k_n$ a set of non-negative integers satisfying $k_1 + k_2 + \cdots + k_n = n$, then we define $P(k_1, \cdots, k_n)$ as the set of permutations of

$$\left\{ \underbrace{1, \cdots, 1}_{k_1}, \underbrace{2, \cdots, 2}_{k_2}, \cdots, \underbrace{n, \cdots, n}_{k_n} \right\}$$

Using (16.70), we have

$$\det(M_n\xi) = \sum_{k_1 + \cdots + k_n = n} \mu(k_1, \cdots, k_n) x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, \tag{16.71}$$

where

$$\mu(k_1, \cdots, k_n) = \sum_{(\alpha_1, \cdots, \alpha_n) \in P(k_1, \cdots, k_n)} \det\left(\begin{bmatrix} W_{11} \cdots W_{1\alpha_1} & \cdots & W_{n\alpha_n} \end{bmatrix} \right).$$

For instance, assume $n = 3$, $k_1 = 0$, $k_2 = 2$, and $k_3 = 1$, then

$$P(k_1,k_2,k_3) = Perm\{2,2,3\} = \{(2,2,3),(2,3,2),(3,2,2)\}.$$

where $Perm\{S\}$ is the set of all permutations of S.

It follows that the coefficient of monomial $x_2^2 x_3$ is

$$\begin{aligned}\mu(0,2,3) &= \det\left(\left[W_{21}\ W_{22}\ W_{33}\right]\right) + \det\left(\left[W_{21}\ W_{32}\ W_{23}\right]\right)\\ &\quad + \det\left(\left[W_{31}\ W_{22}\ W_{2,3}\right]\right).\end{aligned}$$

16.3 Contraction of Tensor Field

This section considers the contraction of tensor fields. The contraction of tensor fields plays a key role in Physics, particularly in Relativity [3, 7].

In Physics, multi-indexes are often used. It is said in [3] that "It is also sometimes convenient to use matrix methods to handle the summations over repeated suffixes. These methods are restricted to quantities carrying either one or two suffixes. enabling them to be arranged as either one-dimensional arrays (row vectors or column vectors) or two-dimensional arrays (matrices)." From this statement one sees that conventional matrix product can be used only for one or two-dimensional data. Using semi-tensor product, the tensor field, which is used to deal with multiple dimensional data, can be treated in matrix form via matrix calculations. Contraction of tensor fields is a perfect example for this.

First, we review the structure matrix of a tensor. Let $\sigma \in T_s^r(V)$ be an (r,s)-type tensor on an n-dimensional vector space V, and $1 \le p \le r$, $1 \le q \le s$. We define the contraction $\pi_q^p : T_s^r(V) \to T_{s-1}^{r-1}(V)$ as follows: Fix the basis of V as $\{d_1,\cdots,d_n\}$ and its dual basis $\{e^1,\cdots,e^n\}$ of V^*. For $\sigma \in T_s^r(V)$ define

$$\omega_{j_1\cdots j_s}^{i_1\cdots i_r} = \sigma(d_{i_1},\cdots,d_{i_r};e^{j_1},\cdots,e^{j_s}), \quad i_1,\cdots,i_r,j_1,\cdots,j_s = 1,\cdots,n.$$

According to this definition, we can obtain a set of n^{r+s} data. They can be arranged into a matrix as

$$M_\sigma = \begin{bmatrix} \omega_{1\cdots 11}^{1\cdots 11} & \omega_{1\cdots 11}^{1\cdots 12} & \cdots & \omega_{1\cdots 11}^{n\cdots nn} \\ \omega_{1\cdots 12}^{1\cdots 11} & \omega_{1\cdots 12}^{1\cdots 12} & \cdots & \omega_{1\cdots 12}^{n\cdots nn} \\ \vdots & \vdots & & \vdots \\ \omega_{n\cdots nn}^{1\cdots 11} & \omega_{n\cdots nn}^{1\cdots 12} & \cdots & \omega_{n\cdots nn}^{n\cdots nn} \end{bmatrix}, \tag{16.72}$$

which is called the structure matrix of σ.

Using structure matrix, we have

$$\begin{aligned}&\sigma(\sigma_1,\cdots,\sigma_s;X_1,\cdots,X_r)\\ &\quad = \sigma_s \ltimes \cdots \ltimes \sigma_1 \ltimes M_\sigma \ltimes X_1 \ltimes \cdots \ltimes X_r.\end{aligned} \tag{16.73}$$

In the following we use structure matrix to define the contraction $\pi_q^p(\sigma)$, the elements of its structure matrix are defined as

$$\omega_{j_1\cdots\hat{j}_q\cdots j_s}^{i_1\cdots\hat{i}_p\cdots i_r} = \sum_{i_p=j_q} \omega_{j_1\cdots j_q\cdots j_s}^{i_1\cdots i_p\cdots i_r}, \tag{16.74}$$

where "$\hat{\cdot}$" means the corresponding index is omitted.

Since the definition depends on the choice of basis, we need to prove the definition is well posed, that is, it is independent on the choice of basis. We will first deduce a formula for the structure matrix of the contracted tensor, and then prove that this formula defines a tensor, which is independent of the choice of the coordinates.

To deduce the structure matrix $\pi_q^p(\sigma)$ set $\xi = n^{s-1}$, $\eta = n^{r-1}$, and split the structure matrix M_σ into $\xi \times \eta$ block form as

$$M_\sigma = \begin{bmatrix} M_{11} & \cdots & M_{1\eta} \\ \vdots & & \vdots \\ M_{\xi 1} & \cdots & M_{\xi\eta} \end{bmatrix}, \tag{16.75}$$

where each block M_{ij} is an $n \times n$ matrix.

Assume p and q correspond to last vector and last co-vector arguments, then a straightforward computation shows the following result.

Lemma 16.6. *Assume $p = r$ and $q = s$, then*

$$M_{\pi_s^r(\sigma)} = \begin{bmatrix} \mathrm{tr}(M_{11}) & \cdots & \mathrm{tr}(M_{1\eta}) \\ \vdots & & \vdots \\ \mathrm{tr}(M_{\xi 1}) & \cdots & \mathrm{tr}(M_{\xi\eta}) \end{bmatrix} := TR(M_\sigma). \tag{16.76}$$

Where the operator TR is used to calculate the traces of each blocks.

For general case we need to interchange the indexes p with r, and q with s. Note that the swap of two elements can be realized by a sequence of swaps of two adjacent elements. Based on this consideration, we can use Proposition 2.7 to realize the required reordering. We leave the detailed proof to the reader and present the corresponding structure matrix of σ as

$$\begin{aligned} \tilde{M}_\sigma &= \prod_{t=0}^{s-q-1} (I_{n^{s-2-t}} \otimes W_{[n]} \otimes I_{n^t}) M_\sigma \prod_{t=0}^{r-p-1} (I_{n^{r-2-t}} \otimes W_{[n]} \otimes I_{n^t}) \\ &:= \Pi_1 M_\sigma \Pi_2. \end{aligned} \tag{16.77}$$

Similar to the case of $p = r$ and $q = s$, we replace M_σ by $\tilde{M}_\sigma$, split the later into $\xi \times \eta$ block form, and denote each $n \times n$ blocks by $\tilde{M}_{ij}$. Then we have

Proposition 16.10. *The structure matrix of $\pi_q^p(\sigma)$ is*

$$M_{\pi_q^p(\sigma)} = TR(\tilde{M}_\sigma) = TR(\Pi_1 M_\sigma \Pi_2). \tag{16.78}$$

We give a numerical example to depict the contraction.

Example 16.11. Let $n = 2$, $r = 2$, and $s = 3$. We consider $\pi_1^1(\sigma)$. Denote

$$M_\sigma = \begin{bmatrix} a_{111}^{11} & a_{111}^{12} & a_{111}^{21} & a_{111}^{22} \\ a_{112}^{11} & a_{112}^{12} & a_{112}^{21} & a_{112}^{22} \\ a_{121}^{11} & a_{121}^{12} & a_{121}^{21} & a_{121}^{22} \\ a_{122}^{11} & a_{122}^{12} & a_{122}^{21} & a_{122}^{22} \\ a_{211}^{11} & a_{211}^{12} & a_{211}^{21} & a_{211}^{22} \\ a_{212}^{11} & a_{212}^{12} & a_{212}^{21} & a_{212}^{22} \\ a_{221}^{11} & a_{221}^{12} & a_{221}^{21} & a_{221}^{22} \\ a_{222}^{11} & a_{222}^{12} & a_{222}^{21} & a_{222}^{22} \end{bmatrix}.$$

$$\begin{aligned} \Pi_1 &= \prod_{t=0}^{1} I_{2^{1-t}} \otimes W_{[2]} \otimes I_{2^t} = (I_2 \otimes W_{[2]})(W_{[2]} \otimes I_2) \\ &= \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}. \end{aligned}$$

$$\Pi_2 = W_{[2]} = \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix}.$$

Then

$$\Pi_1 M_\sigma \Pi_2 = \begin{bmatrix} a_{111}^{11} & a_{111}^{21} & a_{111}^{12} & a_{111}^{22} \\ a_{211}^{11} & a_{211}^{21} & a_{211}^{12} & a_{211}^{22} \\ a_{112}^{11} & a_{112}^{21} & a_{112}^{12} & a_{112}^{22} \\ a_{212}^{11} & a_{212}^{21} & a_{212}^{12} & a_{212}^{22} \\ a_{121}^{11} & a_{121}^{21} & a_{121}^{12} & a_{121}^{22} \\ a_{221}^{11} & a_{221}^{21} & a_{221}^{12} & a_{221}^{22} \\ a_{122}^{11} & a_{122}^{21} & a_{122}^{12} & a_{122}^{22} \\ a_{222}^{11} & a_{222}^{21} & a_{222}^{12} & a_{222}^{22} \end{bmatrix}.$$

Using the above form and the Proposition 16.10, we can obtain that

$$M_{\pi_1^1(\sigma)} = \begin{bmatrix} a_{111}^{11} + a_{211}^{21} & a_{111}^{12} + a_{211}^{22} \\ a_{112}^{11} + a_{212}^{21} & a_{112}^{12} + a_{212}^{22} \\ a_{121}^{11} + a_{221}^{21} & a_{121}^{12} + a_{221}^{22} \\ a_{122}^{11} + a_{222}^{21} & a_{122}^{12} + a_{222}^{22} \end{bmatrix}.$$

Finally we prove that the contracted tensor defined by (16.77) is independent of the choice of the coordinates. Assume we have a coordinate transformation $z = z(x)$, and its Jacobi matrix is $J = \frac{\partial z}{\partial x}$.

The following lemma can be verified by straightforward computations.

Lemma 16.7. *1. Assume $P \in \mathcal{M}_{s\times m}$, $Q \in \mathcal{M}_{n\times n}$, and $A_i \in \mathcal{M}_{n\times rn}$, $i = 1,\cdots,m$. Set*

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_m \end{bmatrix} = (P\otimes Q)\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} = (P\otimes Q)A,$$

then

$$\begin{bmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_m \end{bmatrix} = P \ltimes \begin{bmatrix} QA_1 \\ \vdots \\ QA_m \end{bmatrix}.$$

Moreover, we also have

$$TR(\tilde{A}) = P\cdot TR\left(\begin{bmatrix} QA_1 \\ \vdots \\ QA_m \end{bmatrix}\right).$$

2. Let $P \in M_{m\times s}$, $Q \in M_{n\times n}$, and $A_i \in M_{nr\times n}$, $i = 1,\cdots,m$. Assume

$$\tilde{A} = \left[\tilde{A}_1,\cdots,\tilde{A}_m\right] = [A_1,\cdots,A_m](P\otimes Q) = A(P\otimes Q),$$

then we have

$$\left[\tilde{A}_1,\cdots,\tilde{A}_m\right] = [A_1Q,\cdots,A_mQ]P.$$

Moreover, we also have

$$TR(\tilde{A}) = TR(A_1Q,\cdots,A_mQ)P.$$

Now we are ready to prove the key issue: the contraction defined above is properly defined. That is, we have to prove that the definition is independent of the choice of the coordinates.

Theorem 16.6. *The contracted tensor defined by (16.74) is independent of the choice of coordinates.*

Proof. Let

$$M_\sigma = \begin{bmatrix} M_{11} & \cdots & M_{1\eta} \\ \vdots & & \vdots \\ M_{\xi 1} & \cdots & M_{\xi\eta} \end{bmatrix}.$$

Using Proposition 16.10, we have

$$M_{\pi_q^p(\sigma)} = TR(\Pi_1 M_\sigma \Pi_2).$$

Now consider a coordinate change $z = z(x)$. Under this new coordinate frame M_σ becomes

$$\bar{M}_\sigma = \underbrace{J^{-1} \otimes \cdots \otimes J^{-1}}_{s} M_\sigma \underbrace{J \otimes \cdots \otimes J}_{t}.$$

Note that Π_1 is commutative with $\underbrace{J^{-1} \otimes \cdots \otimes J^{-1}}_{s}$ and Π_2 is commutative with $\underbrace{J \otimes \cdots \otimes J}_{t}$. Applying Proposition 16.10 to $\bar{M}_\sigma$ yields

$$\begin{aligned}
\bar{M}_{\pi_q^p(\sigma)} &= TR\left(\Pi_1 (\underbrace{J^{-1} \otimes \cdots \otimes J^{-1}}_{s}) M_\sigma (\underbrace{J \otimes \cdots \otimes J}_{t}) \Pi_2\right) \\
&= TR\left((\underbrace{J^{-1} \otimes \cdots \otimes J^{-1}}_{s})(\Pi_1 M_\sigma \Pi_2)(\underbrace{J \otimes \cdots \otimes J}_{t})\right) \\
&= TR\left(((\underbrace{J^{-1} \otimes \cdots \otimes J^{-1}}_{s-1}) \otimes J^{-1})(\tilde{M}_\sigma)((\underbrace{J \otimes \cdots \otimes J}_{t-1}) \otimes J)\right) \\
&= (\underbrace{J^{-1} \otimes \cdots \otimes J^{-1}}_{s-1}) TR(J^{-1}(\tilde{M}_\sigma) J)(\underbrace{J \otimes \cdots \otimes J}_{t-1}) \\
&= (\underbrace{J^{-1} \otimes \cdots \otimes J^{-1}}_{s-1}) TR(\tilde{M}_\sigma)(\underbrace{J \otimes \cdots \otimes J}_{t-1}) \\
&= (\underbrace{J^{-1} \otimes \cdots \otimes J^{-1}}_{s-1}) M_{\pi_q^p(\sigma)} (\underbrace{J \otimes \cdots \otimes J}_{t-1}).
\end{aligned}$$

Note that the last three equalities are obtained by using Lemma 16.7. □

Exercise 16

1. (to be completed).

References

1. Abraham, R., Marsden, J.: Foundations of Mechanics, 2nd edn. Benjamin/Cummings Pub. Com. Inc. (1978)
2. Boothby, W.: An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd edn. Academic Press, Inc., Orlando (1986)
3. Foster, J., Nightngale, J.: A Short Course in General Relativity. Springer-Verlag (1995)
4. Greub, W.: Linear Algebra, 4th edn. Springer-Verlag (1981)
5. Hartshorne, R.: Algebraic Geometry. Springer-Verlag, New York (1977)
6. Hungerford, T.: Algebra. Springer-Verlag, New York (1974)
7. Sachs, R., Wu, H.: General Relativity for Mathematicians. Springer-Verlag (1977)