Chapter 8
Fuzzy Control

Since Zadeh's pioneering work [3], the fuzzy control has been studied widely for more than forty years. Now it has been developed into a relatively mature theoretical system [4, 1], and received many applications [2].

This chapter introduces fuzzy control. We first introduce the fuzzy relations. Particularly, we emphasize multiple fuzzy relations.

8.1 Fuzzy Relations

We first introduce the fuzzy relations between two universes of discourse, which is commonly used in fuzzy control.

Definition 8.1. 1. Let E and F be two sets. The product set $E \times F$ is defined as the set of pairs as

$$E \times F = \{(e,d) | e \in E \text{ and } d \in F\}.$$

2. A fuzzy relation between E and F is a fuzzy set $R \in \mathscr{F}(E \times F)$.

Assume $E = \{e_1, \cdots, e_m\}$ and $F = \{d_1, \cdots, d_n\}$. Then a fuzzy relation R between E and F can be expressed by a matrix, called the matrix form of relation R, defined as

$$M_R = \begin{bmatrix} \mu_R(e_1,d_1) & R(e_1,d_2) & \cdots & \mu_R(e_1,d_n) \\ \mu_R(e_2,d_1) & R(e_2,d_2) & \cdots & \mu_R(e_2,d_n) \\ \vdots & & & \\ \mu_R(e_m,d_1) & R(e_m,d_2) & \cdots & \mu_R(e_m,d_n) \end{bmatrix} \tag{8.1}$$

Remark 8.1. 1. Since $M_R \in \mathscr{B}^k_{m \times n}$, the matrices of relations satisfy Proposition **??**.
2. Since M_R is essentially the vector form of the fuzzy set R (by identifying M_R with $V_r(M_R)$ or $V_c(M_R)$), Proposition **??** and Remark **??** hold. Particularly, equalities (7.41)–(7.45) are still correct.

Definition 8.2. 1. $R \in \mathscr{F}(E \times F)$ is called a zero relation, if $\mu_R(x,y) = 0$, $\forall x \in E, \forall y \in F$.
2. $R \in \mathscr{F}(E \times F)$ is called a universal relation, if $\mu_R(x,y) = 1, \forall x \in E, \forall y \in F$.
3. Let $R \in \mathscr{F}(E \times F)$, $S \in \mathscr{F}(F \times E)$ is called the inverse relation of R. If

$$\mu_R(x,y) = \mu_S(y,x), \quad \forall x \in E, \forall y \in F.$$

As a convention, the inverse of R is denoted by R^T. It follows from the definition that

$$M_{R^T} = (M_R)^T. \tag{8.2}$$

The inverse relation has the following properties.

Proposition 8.1. *Assume R, T, $R_\lambda \in \mathscr{F}(E \times F)$, $\lambda \in \Lambda$. Then*

1.

$$(R^T)^c = (R^c)^T. \tag{8.3}$$

2.

$$(\cup_{\lambda \in \Lambda} R_\lambda)^T = \cup_{\lambda \in \Lambda} (R_\lambda)^T. \tag{8.4}$$

3.

$$(\cap_{\lambda \in \Lambda} R_\lambda)^T = \cap_{\lambda \in \Lambda} (R_\lambda)^T. \tag{8.5}$$

4. If $R \subset T$, then

$$R^T \subset T^T.$$

Next, we consider the compounded relations.

Definition 8.3. Let E, F, G be three sets, R and S be two relations over $E \times F$ and $F \times G$ respectively. That is, $R \in \mathscr{F}(E \times F)$, $S \in \mathscr{F}(F \times G)$. Then the compounded relation $R \circ S \in \mathscr{F}(E \times G)$ is a relation on $E \times G$, defined as

$$\mu_{R \circ S}(e,g) = \vee_{d \in F} \left[\mu_R(e,d) \wedge \mu_S(d,g) \right], \quad e \in E, \ g \in G.$$

The following result is an immediate consequence of the definition.

Proposition 8.2. *Let E, F, and G be three finite sets, and $R \in \mathscr{F}(E \times F)$, $S \in \mathscr{F}(F \times G)$. Assume R and S have their matrix forms as M_R and M_S respectively. Then*

$$M_{R \circ S} = M_R \times_{\mathscr{B}} M_S. \tag{8.6}$$

Consider the relations on same universe.

Definition 8.4. 1. $R \in \mathscr{F}(E \times E)$ is called an identity relation, if

$$\mu_R(x,y) = \begin{cases} 1, & x = y \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $|E| = n$, then identity relation R has its matrix form $M_R = I_n$.

2. $R \in \mathscr{F}(E \times E)$ is said to be self-related, if

$$\mu_R(x,x) = 1, \quad \forall x \in E.$$

It is said to be self-unrelated, if

$$\mu_R(x,x) = 0, \quad \forall x \in E.$$

3. $R \in \mathscr{F}(E \times E)$ is said to be symmetric, if

$$\mu_R(x,y) = \mu_R(y,x), \quad \forall x,y \in E.$$

4. $R \in \mathscr{F}(E \times E)$ is said to be transitive, if

$$R \times_{\mathscr{B}} R = R^{2\mathscr{B}} \subset R.$$

A relation may connect more than two sets. In the following we consider a relation on three sets. More than three cases can be treated in exactly the same way.

Definition 8.5. Let X, Y and Z be three sets. A relation among them is a fuzzy set $R \in \mathscr{F}(X \times Y \times Z)$.

Assume $X = \{x_1, x_2, \cdots, x_m\}$, $Y = \{y_1, y_2, \cdots, y_n\}$, and $Z = \{z_1, z_2, \cdots, z_r\}$. Then we can arrange $\{\mu_R(x_i, y_j, z_k) | i = 1, \cdots, m; j = 1, \cdots, n; k = 1, \cdots, r\}$ as a relation matrix. There are several ways to do this. If we arrange it using row order $Id(i;m)$ and column order $Id(j,k;n,r)$, then we have

$$M_{R(X \times YZ)} = \begin{bmatrix} \mu_A(x_1,y_1,z_1) & \cdots & \mu_A(x_1,y_1,z_r) & \cdots & \mu_A(x_1,y_n,z_1) & \cdots & \mu_A(x_1,y_n,z_r) \\ \mu_A(x_2,y_1,z_1) & \cdots & \mu_A(x_2,y_1,z_r) & \cdots & \mu_A(x_2,y_n,z_1) & \cdots & \mu_A(x_2,y_n,z_r) \\ \vdots & & & & & & \\ \mu_A(x_m,y_1,z_1) & \cdots & \mu_A(x_m,y_1,z_r) & \cdots & \mu_A(x_m,y_n,z_1) & \cdots & \mu_A(x_m,y_n,z_r) \end{bmatrix}. \tag{8.7}$$

If we use row order $Id(j;n)$ and column order $Id(i,k;m,r)$, then we have

$$M_{R(Y \times XZ)} = \begin{bmatrix} \mu_A(x_1,y_1,z_1) & \cdots & \mu_A(x_1,y_1,z_r) & \cdots & \mu_A(x_m,y_1,z_1) & \cdots & \mu_A(x_m,y_1,z_r) \\ \mu_A(x_1,y_2,z_1) & \cdots & \mu_A(x_1,y_2,z_r) & \cdots & \mu_A(x_m,y_2,z_1) & \cdots & \mu_A(x_m,y_2,z_r) \\ \vdots & & & & & & \\ \mu_A(x_1,y_n,z_1) & \cdots & \mu_A(x_1,y_n,z_r) & \cdots & \mu_A(x_m,y_n,z_1) & \cdots & \mu_A(x_m,y_n,z_r) \end{bmatrix}. \tag{8.8}$$

Definition 8.6. Let X, Y and Z be three sets.

1. Assume there are two relations as $R \in \mathscr{F}(X \times Z)$ and $S \in \mathscr{F}(Y \times Z)$. Then we define a mapping $* : \mathscr{F}(X \times Z) \times \mathscr{F}(Y \times Z) \to \mathscr{F}(X \times Y \times Z)$ as

$$\mu_{R*S}(x,y,z) = \mu_R(x,z) \wedge \mu_S(y,z), \quad x \in X,\ y \in Y,\ z \in Z.$$

2. Assume there are two relations as $R \in \mathscr{F}(X \times Y)$ and $S \in \mathscr{F}(X \times Z)$. Then we define a mapping $* : \mathscr{F}(X \times Y) \times \mathscr{F}(X \times Z) \to \mathscr{F}(X \times Y \times Z)$ as

$$R * S = \left[R^T * S^T\right]^T.$$

The following result is an immediate consequence of the definition.

Proposition 8.3. *Let $X = \{x_1, x_2, \cdots, x_m\}$, $Y = \{y_1, y_2, \cdots, y_n\}$, and $Z = \{z_1, z_2, \cdots, z_r\}$.*

1. *Assume there are two relations $R \in \mathscr{F}(X \times Z)$ and $S \in \mathscr{F}(Y \times Z)$ with their matrix form as M_R and M_S respectively. Then*

$$M_{R*S(XY \times Z)} = M_R *_{\mathscr{B}} M_S. \tag{8.9}$$

2. *Assume there are two relations $R \in \mathscr{F}(X \times Y)$ and $S \in \mathscr{F}(X \times Z)$ with their matrix form as M_R and M_S respectively. Then*

$$M_{R*S(X \times YZ)} = \left(M_R^T *_{\mathscr{B}} M_S^T\right)^T. \tag{8.10}$$

The compounded relation can be obtained via multi-relations. Principally, they are the same. We give an example to show this.

Example 8.1. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2\}$, and $W = \{w_1, w_2, w_4\}$. We have $R \in \mathscr{F}(X \times Y \times Z)$, $S \in \mathscr{F}(Y \times W)$, and $T \in \mathscr{F}(Z \times W)$. Their matrix forms are

$$M_{R(X \times YZ)} = \begin{bmatrix} 0.2 & 0 & 0.1 & 0.5 & 0.9 & 1 \\ 0.4 & 0.3 & 0.7 & 0.8 & 0 & 0 \end{bmatrix};$$

$$M_S = \begin{bmatrix} 0 & 0.5 & 0.6 \\ 0.1 & 0.3 & 0.8 \\ 0.2 & 0.7 & 1 \end{bmatrix}; \quad M_T = \begin{bmatrix} 0 & 0.1 & 0.9 \\ 0.5 & 1 & 0.3 \end{bmatrix}.$$

Intuitively, W has relation with X through both Y and Z. Therefore, we intend to find the relation of X and W. First, we calculate product M_S with M_T to get a relation in $Y \times Z \times W$ as

$$M_{S*T(YZ \times W)} = M_S * M_T = \begin{bmatrix} 0 & 0.1 & 0.6 \\ 0 & 0.5 & 0.3 \\ 0 & 0.1 & 0.8 \\ 0.1 & 0.3 & 0.3 \\ 0 & 0.1 & 0.9 \\ 0.2 & 0.7 & 0.3 \end{bmatrix}.$$

Then $\Psi = R \circ (S * T) \in \mathscr{F}(X, W)$ has its matrix form as

$$\Psi = M_{R(X\times YZ)} \times_{\mathscr{B}} M_{S*T(YZ\times W)} = \begin{bmatrix} 0.2 & 0.7 & 0.9 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}.$$

8.2 Fuzzy Inference

Definition 8.7. Let $E = \{e_\xi | \xi \in \Xi\}$ and $F = \{f_\lambda | \lambda \in \Lambda\}$ be two universes of discourse. $R \in \mathscr{F}(E \times F)$. Then R determined a mapping $\pi_R : \mathscr{F}(F) \to \mathscr{F}(E)$ as follows: Let $B \in \mathscr{F}(F)$ and $A = \pi_R(B) \in \mathscr{F}(E)$, which is defined as

$$\mu_A(e_\xi) = \vee_{\lambda\in\Lambda} \mu_R(e_\xi, f_\lambda) \wedge \mu_B(\lambda), \quad \xi \in \Xi. \tag{8.11}$$

π_R is called a fuzzy inference (based on R).

Assume $E = \{e_1, \cdots, e_m\}$, $F = \{d_1, \cdots, d_n\}$, and $R \in \mathscr{F}(E \times F)$ has its structure matrix $M_R = (r_{i,j}) \in \mathscr{B}^\infty_{m\times n}$, where

$$r_{i,j} = \mu_R(e_i, d_j), \quad i = 1, \cdots, m; j = 1, \cdots, n.$$

The following result comes from definition immediately. In fact, it is commonly used in fuzzy control.

Proposition 8.4. *Let $R \in \mathscr{F}(E \times F)$ be given as in the above. Assume $\pi_R(B) = A$, where $A \in \mathscr{F}(E)$ and $B \in \mathscr{F}(F)$ with their vector forms $X_A \in \mathscr{B}^\infty_{m\times 1}$ and $X_B \in \mathscr{B}^\infty_{n\times 1}$ respectively. Then*

$$X_A = M_R \times_{\mathscr{B}} X_B. \tag{8.12}$$

Next, we extend it to multiple relation case. Assume there is a fuzzy relation $R \in \mathscr{F}(\prod_{i=1}^k E_i)$, where $E_i = \{e^i_1, \cdots, e^i_{n_i}\}$, $i = 1, \cdots, k$. Moreover, we have $k-1$ fuzzy sets in any $k-1$ factor spaces. Then the fuzzy inference is to produce a fuzzy set in the remain factor space. For statement ease, assume $A_i \in \mathscr{F}(E_i)$, $i = 2, \cdots, k$ are given. Then $\pi_R : \prod_{i=2}^k \mathscr{F}(E_i) \to \mathscr{F} E_1$ is defined by

$$X_{A_1} = M_R \ltimes_{\mathscr{B}} X_{A_2} \ltimes_{\mathscr{B}} \cdots \ltimes_{\mathscr{B}} X_{A_k}, \tag{8.13}$$

where $M_R = M_{E_1\times E_2 E_3\cdots E_k}$ is a matrix of $\{r_{i_1,\cdots,i_k}\}$, arranged in the order of $\mathrm{id}(i_1; n_1) \times \mathrm{id}(i_2, \cdots, i_k; n_2, \cdots, n_k)$.

Note that the arrangement of the entries in M_R depends on the unknown fuzzy set.

Now assume $\{1, \cdots, k\} = \mathscr{I} \cup \mathscr{J}$, where $\mathscr{I} \cup \mathscr{J}$ form a partition of the index set. Assume A_i, $i \in \mathscr{I}$ is a given fuzzy sets. Then using relation R, we can only deduce a fuzzy relation $R' \in \mathscr{F}(\prod_{j\in\mathscr{J}} E_j)$.

We give an example to show how to do this.

Example 8.2. Assume we have universes of discourse $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3\}$, and $Z = \{z_1, z_2\}$. Moreover, a relation $R \in \mathscr{F}(X \times Y \times Z)$ is given, i.e.,

$$r_{i,j,k} = \mu_R(x_i, y_j, z_k), \quad i = 1,2,3,4;\ j = 1,2,3;\ k = 1,2,$$

are known.

Now assume we have $A \in \mathscr{F}(X)$ and $B \in \mathscr{F}(Y)$. Using R, we can have a fuzzy inference $C \in \mathscr{Z}$. Then following is a numerical example.

Assume M_R is arranged as $\mathrm{id}(k;2) \times \mathrm{id}(i,j;4,3)$ as

$$M_R = \begin{bmatrix} 0 & 0.3 & 0.7 & 1 & 0.5 & 0.9 & 0.4 & 0.1 & 0 & 0.3 & 0.1 & 0 \\ 1 & 0.2 & 0.3 & 0.5 & 0.3 & 0.2 & 0.7 & 0.8 & 1 & 0.4 & 0.6 & 1 \end{bmatrix},$$

and the vector forms of A and B are

$$A = [0.1\ 0.5\ 1\ 0.4]^T;$$
$$B = [0.8\ 0.7\ 0.5]^T.$$

Then we have C as

$$C = M_R \ltimes_{\mathscr{B}} A \ltimes_{\mathscr{B}} = [0.5\ 0.7]^T.$$

When only a fuzzy set $A \in \mathscr{F}(E)$ is given. Then the fuzzy inference provides a relation $R' \in \mathscr{F}(F,G)$, which has the matrix form as

$$M_{R'} = M_R \ltimes A.$$

If we use previous R and A, then we have

$$M_{R'} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.7 & 0.8 & 1 \end{bmatrix}.$$

Next, we consider the compounded relations. Assume we have two fuzzy relations $R \in \mathscr{F}(E \times F)$ and $S \in \mathscr{F}(F \times G)$. Then $T := R \circ S$ is a relation in $\mathscr{F}(E \times G)$ with its matrix form as

$$M_T = M_R \circ M_S. \tag{8.14}$$

We are particularly interested in the composition of multiple fuzzy relations.

Definition 8.8. Let E, F, G be three sets, R and S be two relations over $E \times F$ and $F \times G$ respectively. That is, $R \in \mathscr{F}(E \times F)$ $S \in \mathscr{F}(F \times G)$. Then the compounded relation $R \circ S \in \mathscr{F}(E \times G)$ is a relation on $E \times G$, defined as

$$\mu_{R\circ S}(e,g) = \vee_{d\in F}\left[\mu_R(e,d) \wedge \mu_S(d,g)\right], \quad e \in E,\ g \in G.$$

The following result is an immediate consequence of the definition.

Proposition 8.5. *Let E, F, and G be three finite sets, and $R \in \mathscr{F}(E \times F)$, $S \in \mathscr{F}(F \times G)$. Assume R and S have their matrix forms as M_R and M_S respectively. Then*

$$M_{R\circ S} = M_R \times_{\mathscr{B}} M_S. \tag{8.15}$$

Consider the relations on same universe.

Definition 8.9. 1. $R \in \mathscr{F}(E \times E)$ is called an identity relation, if

$$\mu_R(x,y) = \begin{cases} 1, & x = y \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $|E| = n$, then identity relation R has its matrix form $M_R = I_n$.

2. $R \in \mathscr{F}(E \times E)$ is said to be self-related, if

$$\mu_R(x,x) = 1, \quad \forall x \in E.$$

It is said to be self-unrelated, if

$$\mu_R(x,x) = 0, \quad \forall x \in E.$$

3. $R \in \mathscr{F}(E \times E)$ is said to be symmetric, if

$$\mu_R(x,y) = \mu_R(y,x), \quad \forall x,y \in E.$$

4. $R \in \mathscr{F}(E \times E)$ is said to be transitive, if

$$R \times_{\mathscr{B}} R = R^{2_{\mathscr{B}}} \subset R.$$

A relation may connect more than two sets. In the following we consider a relation on three sets. More than three cases can be treated in exactly the same way.

Definition 8.10. Let X, Y and Z be three sets. A relation among them is a fuzzy set $R \in \mathscr{F}(X \times Y \times Z)$.

Assume $X = \{x_1, x_2, \cdots, x_m\}$, $Y = \{y_1, y_2, \cdots, y_n\}$, and $Z = \{z_1, z_2, \cdots, z_r\}$. Then we can arrange $\{\mu_R(x_i, y_j, z_k) | i = 1, \cdots, m; j = 1, \cdots, n; k = 1, \cdots, r\}$ as a relation matrix. There are several ways to do this. If we arrange it using row order $Id(i;m)$ and column order $Id(j,k;n,r)$, then we have

$$M_{R(X \times YZ)} = \begin{bmatrix} \mu_A(x_1,y_1,z_1) & \cdots & \mu_A(x_1,y_1,z_r) & \cdots & \mu_A(x_1,y_n,z_1) & \cdots & \mu_A(x_1,y_n,z_r) \\ \mu_A(x_2,y_1,z_1) & \cdots & \mu_A(x_2,y_1,z_r) & \cdots & \mu_A(x_2,y_n,z_1) & \cdots & \mu_A(x_2,y_n,z_r) \\ \vdots & & & & & & \\ \mu_A(x_m,y_1,z_1) & \cdots & \mu_A(x_m,y_1,z_r) & \cdots & \mu_A(x_m,y_n,z_1) & \cdots & \mu_A(x_m,y_n,z_r) \end{bmatrix}. \tag{8.16}$$

If we use row order $Id(j;n)$ and column order $Id(i,k;m,r)$, then we have

$$M_{R(Y\times XZ)} = \begin{bmatrix} \mu_A(x_1,y_1,z_1) & \cdots & \mu_A(x_1,y_1,z_r) & \cdots & \mu_A(x_m,y_1,z_1) & \cdots & \mu_A(x_m,y_1,z_r) \\ \mu_A(x_1,y_2,z_1) & \cdots & \mu_A(x_1,y_2,z_r) & \cdots & \mu_A(x_m,y_2,z_1) & \cdots & \mu_A(x_m,y_2,z_r) \\ \vdots \\ \mu_A(x_1,y_n,z_1) & \cdots & \mu_A(x_1,y_n,z_r) & \cdots & \mu_A(x_m,y_n,z_1) & \cdots & \mu_A(x_m,y_n,z_r) \end{bmatrix}. \tag{8.17}$$

Definition 8.11. Let X, Y and Z be three sets.

1. Assume there are two relations as $R \in \mathscr{F}(X \times Z)$ and $S \in \mathscr{F}(Y \times Z)$. Then we define a mapping $* : \mathscr{F}(X \times Z) \times \mathscr{F}(Y \times Z) \to \mathscr{F}(X \times Y \times Z)$ as

$$\mu_{R*S}(x,y,z) = \mu_R(x,z) \wedge \mu_S(y,z), \quad x \in X,\ y \in Y,\ z \in Z.$$

2. Assume there are two relations as $R \in \mathscr{F}(X \times Y)$ and $S \in \mathscr{F}(X \times Z)$. Then we define a mapping $* : \mathscr{F}(X \times Y) \times \mathscr{F}(X \times Z) \to \mathscr{F}(X \times Y \times Z)$ as

$$R * S = \left[R^T * S^T\right]^T.$$

The following result is an immediate consequence of the definition.

Proposition 8.6. *Let* $X = \{x_1, x_2, \cdots, x_m\}$, $Y = \{y_1, y_2, \cdots, y_n\}$, *and* $Z = \{z_1, z_2, \cdots, z_r\}$.

1. *Assume there are two relations* $R \in \mathscr{F}(X \times Z)$ *and* $S \in \mathscr{F}(Y \times Z)$ *with their matrix form as* M_R *and* M_S *respectively. Then*

$$M_{R*S(XY\times Z)} = M_R *_{\mathscr{B}} M_S. \tag{8.18}$$

2. *Assume there are two relations* $R \in \mathscr{F}(X \times Y)$ *and* $S \in \mathscr{F}(X \times Z)$ *with their matrix form as* M_R *and* M_S *respectively. Then*

$$M_{R*S(X\times YZ)} = \left(M_R^T *_{\mathscr{B}} M_S^T\right)^T. \tag{8.19}$$

The compounded relation can be obtained via multi-relations. Principally, they are the same. We give an example to show this.

Example 8.3. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2\}$, and $W = \{w_1, w_2, w_4\}$. We have $R \in \mathscr{F}(X \times Y \times Z)$, $S \in \mathscr{F}(Y \times W)$, and $T \in \mathscr{F}(Z \times W)$. Their matrix forms are

$$M_{R(X\times YZ)} = \begin{bmatrix} 0.2 & 0 & 0.1 & 0.5 & 0.9 & 1 \\ 0.4 & 0.3 & 0.7 & 0.8 & 0 & 0 \end{bmatrix};$$

$$M_S = \begin{bmatrix} 0 & 0.5 & 0.6 \\ 0.1 & 0.3 & 0.8 \\ 0.2 & 0.7 & 1 \end{bmatrix}; \quad M_T = \begin{bmatrix} 0 & 0.1 & 0.9 \\ 0.5 & 1 & 0.3 \end{bmatrix}.$$

Intuitively, W has relation with X through both Y and Z. Therefore, we intend to find the relation of X and W. First, we calculate product M_S with M_T to get a relation in $Y \times Z \times W$ as

$$M_{S*T(YZ\times W)} = M_S * M_T = \begin{bmatrix} 0 & 0.1 & 0.6 \\ 0 & 0.5 & 0.3 \\ 0 & 0.1 & 0.8 \\ 0.1 & 0.3 & 0.3 \\ 0 & 0.1 & 0.9 \\ 0.2 & 0.7 & 0.3 \end{bmatrix}.$$

Then $\Psi = R \circ (S * T) \in \mathscr{F}(X, W)$ has its matrix form as

$$\Psi = M_{R(X\times YZ)} \times_{\mathscr{B}} M_{S*T(YZ\times W)} = \begin{bmatrix} 0.2 & 0.7 & 0.9 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}.$$

8.3 Fuzzy Control

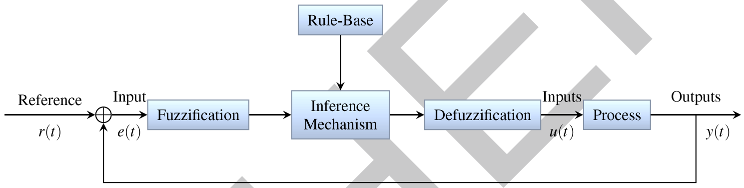

Fig. 8.1 A fuzzy control system

Fig. 8.1 [1] shows the structure of a fuzzy control system. In this section, a general framework, based on matrix approach, is investigated.

8.3.1 Fuzzification

This section considers only the fuzzification. We first introduce a dual fuzzy structure.

Definition 8.12. 1. Let E be a universe of discourse, and $\mathscr{A} = \{A_1, \cdots, A_k\}$ be a set of fuzzy sets on E. Then $(E, \mathscr{A})$ is called a fuzzy structure. The support of A_i is defined as

$$\mathrm{Supp}(A_i) = \{e \in E | \mu_{A_i}(e) \neq 0\} \subset E.$$

2. Assume E is a well ordered set. $\mathscr{A} = \{A_1, \cdots, A_k\}$ is called a set of degree-based fuzzy sets, if

$$\sup(\text{supp}(A_i)) < \sup(\text{supp}(A_{i+1})), \quad \text{and} \quad \inf(\text{supp}(A_i)) < \inf(\text{supp}(A_{i+1})),$$
$$i = 1, \cdots, k-1. \tag{8.20}$$

3. Given a fuzzy structure $(E, \mathscr{A})$ as in item 1. Assume E is a well ordered set and $\mathscr{A}$ is a set of degree-based sets, i.e., (8.20) is satisfied. Then we may consider $(\mathscr{A}, E)$ as a fuzzy structure, where $\mathscr{A} = \{A_1, \cdots, A_k\}$ is considered as a unverse of discourse, each $e \in E$ is a fuzzy set, with

$$\mu_e(A_i) := \mu_{A_i}(e), \quad i = 1, \cdots, k. \tag{8.21}$$

This fuzzy structure is called the dual structure of $(E, \mathscr{A})$.

In fact, fuzzification is basically finding the dual structure. We use an example to describe this.

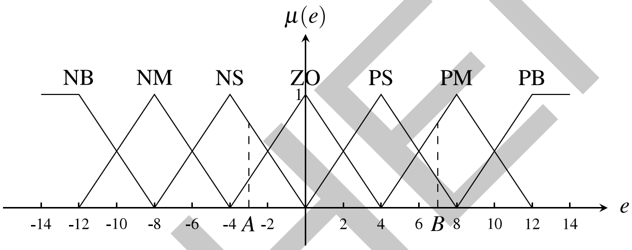

Fig. 8.2 The membership functions of fuzzy set

Example 8.4. Consider a measurement error e, which could be $[-14, 14]$. We consider 7 fuzzy sets, which have linguistic statements respectively as: NB (negative big), NM (negative medium), NS (negative small), ZO (Zero), PS (positive small), PM (positive medium), and PB (positive big). The membership functions of the fuzzy sets are depicted in Fig. 8.2. Now for each point x in the universe of discourse $E = [-14, 14]$, we know its membership degrees for each fuzzy set. For instance, for point A, we have

$$\mu_{NS}(A) = 0.75;\ \mu_{ZO}(A) = 0.25;\ \mu_Y(A) = 0,\ Y = NB, NM, PS, PM, PB. \tag{8.22}$$

Similarly, for point B we have

$$\mu_{PS}(B) = 0.25;\ \mu_{PM}(B) = 0.75;\ \mu_Y(A) = 0,\ Y = NB, NM, NS, ZO, PB. \tag{8.23}$$

Checking the fuzzification in fuzzy control, one sees easily that what we need to do is to convert a point $e \in E$ to a fuzzy set. To do this, people basically interchange the universe of discourse with the set of degree-based fuzzy sets. Precisely, as in Example **??** we consider

$$D := \{d_1 = NB,\ d_2 = NM,\ d_3 = NS,\ d_4 = ZO,\ d_5 = PS,\ d_6 = PM,\ d_7 = PB\}$$

as the universe of intercourse and consider each $e \in E$ as a fuzzy set. In this consideration we can express (8.22) and (8.23) respectively as

$$\mu_A(e_3) = 0.75;\ \mu_A(e_4) = 0.25;\ \mu_A(e_i) = 0,\ i = 1,2,5,6,7. \tag{8.24}$$

$$\mu_B(e_5) = 0.25;\ \mu_B(e_6) = 0.75;\ \mu_B(e_i) = 0,\ i = 1,2,3,4,7. \tag{8.25}$$

In vector form we have

$$\begin{aligned} A &= [0\ 0\ 0.75\ 0.25\ 0\ 0\ 0]^T; \\ B &= [0\ 0\ 0\ 0\ 0.25\ 0.75\ 0]^T. \end{aligned} \tag{8.26}$$

8.3.2 Fuzzy Controller

In general, a fuzzy controller is a fuzzy inference mechanism. In general, assume the system in Fig. 8.1 has m inputs and p outputs, then the fuzzy controller has the form as

$$\Sigma \in \mathscr{F}(Y_1 \times \cdots \times Y_p \times U_1 \times \cdots \times U_m), \tag{8.27}$$

where Y_i, $i = 1,\cdots,p$, and U_j, $j = 1,\cdots,m$ have been fuzzificated and described in the previous subsection.

We give an example to depict this.

Example 8.5 ([5]). Assume a system has a single control U, which depends on A and B. Both A and B have 7 levels as $\{NB,\ NM,\ NS,\ ZO,\ PS,\ PM, PB\}$, and U has 13 levels as $\{NVB,\ NB,\ NMB,\ NMS,\ NS,\ NVS,\ ZO,\ PVS,\ PS,\ PMS,\ PMB,\ PB,\ PVB\}$. Using the previous orders, we denote by

$$\begin{aligned} E_A &= \{a_1,\cdots,a_7\}, \\ E_B &= \{b_1,\cdots,b_7\}, \\ E_U &= \{u_1,\cdots,u_{13}.\} \end{aligned}$$

the universes of discourse for A, B, and U respectively.

1. In general, we can arrange

$$\left\{ \mu_\Sigma(a_i,b_j,c_k) \,\middle|\, i = 1,\cdots,7;\ j = 1,\cdots,7;\ k = 1,\cdots,13 \right\}$$

into a matrix, M_Σ, in the order of $\mathrm{id}(k;13) \times \mathrm{id}(i,j;7,7)$. Then

$$M_\Sigma \in \mathscr{B}^\infty_{13\times 7^2}. \tag{8.28}$$

2. Particularly, we may use the "If $A = \times$, and $B = \times$, Then $U = \times$" rules, a rule table can be obtained. For instance, we have Table 8.1.

Table 8.1 Rule Table

$A\backslash U\backslash B$	NB	NM	NS	ZO	PS	PM	PN
NB	-1	-0.8	-0.6	-0.4	-0.2	-0.1	0
NM	-0.8	-0.6	-0.4	-0.2	-0.1	0	0.1
NS	-0.6	-0.4	-0.2	-0.1	0	0.1	0.2
ZO	-0.4	-0.2	-0.1	0	0.1	0.2	0.4
PS	-0.2	-0.1	0	0.1	0.2	0.4	0.6
PM	-0.1	0	0.1	0.2	0.4	0.6	0.8
PB	0	0.1	0.2	0.4	0.6	0.8	1

To use vector expression we identify

$$\begin{array}{l} NB \sim \delta_7^7;\ NM \sim \delta_7^6;\ \cdots\ PB \sim \delta_7^1; \\ -1 \sim \delta_{13}^{13};\ -0.8 \sim \delta_{13}^{12};\ \cdots\ 1 \sim \delta_{13}^1. \end{array}$$

Then M_Σ can be expressed as

$$\begin{array}{rl} M_\Sigma = \delta_{13}[& 13\ 12\ 11\ 10\ 9\ 8\ 7\ 6\ 12\ 11\ 10\ 9\ 8\ 7\ 6\ 5 \\ & 11\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3 \\ & 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \\ & 7\ 6\ 5\ 4\ 3\ 2\ 1\ 0] \in \mathscr{B}_{13\times 49}^2. \end{array} \tag{8.29}$$

It is easy to see that (8.29), which is obtained by "If... Then..." rules, is a particular case of (8.27).

In general case a fuzzy controller is mathematically equivalent to a fuzzy relation (8.27). To get its matrix expression, we specify Y_i and U_j as

$$\begin{array}{ll} Y_i = \left\{y_1^i, \cdots, y_{\alpha_i}^i\right\}, & i = 1, \cdots, p; \\ U_j = \left\{u_1^j, \cdots, u_{\beta_j}^j\right\}, & j = 1, \cdots, m. \end{array} \tag{8.30}$$

Then we have

$$\begin{array}{l} \mu_\Sigma\left(y_{\xi_1}^1, \cdots, y_{\xi_p}^p, u_{\eta_1}^1, \cdots, u_{\eta_m}^m\right) := \gamma_{\eta_1\cdots\eta_m}^{\xi_1\cdots\xi_p}, \\ \xi_i = 1, \cdots, \alpha_i,\ i = 1, \cdots, p;\ \eta_j = 1, \cdots, \beta_j,\ j = 1, \cdots, m. \end{array} \tag{8.31}$$

Arranging $\left\{\gamma_{\eta_1\cdots\eta_m}^{\xi_1\cdots\xi_p}\right\}$ into a matrix in the order of $\mathrm{id}(\eta_1, \cdots, \eta_m; \beta_1, \cdots, \beta_m) \times \mathrm{id}(\xi_1, \cdots, \xi_p; \alpha_1, \cdots, \alpha_p)$, we have

$$M_\Sigma = \begin{bmatrix} \gamma_{1\cdots 11}^{1\cdots 11} & \gamma_{1\cdots 11}^{1\cdots 12} & \cdots & \gamma_{1\cdots 11}^{1\cdots 1\alpha_p} & \cdots & \gamma_{1\cdots 11}^{\alpha_1\cdots\alpha_p} \\ \gamma_{1\cdots 12}^{1\cdots 11} & \gamma_{1\cdots 12}^{1\cdots 12} & \cdots & \gamma_{1\cdots 12}^{1\cdots 1\alpha_p} & \cdots & \gamma_{1\cdots 12}^{\alpha_1\cdots\alpha_p} \\ \vdots \\ \gamma_{\beta_1\cdots\beta_m}^{1\cdots 11} & \gamma_{\beta_1\cdots\beta_m}^{1\cdots 12} & \cdots & \gamma_{\beta_1\cdots\beta_m}^{1\cdots 1\alpha_p} & \cdots & \gamma_{\beta_1\cdots\beta_m}^{\alpha_1\cdots\alpha_p} \end{bmatrix} \tag{8.32}$$

Now assume a fuzzy controller is designed, which means the structure matrix M_Σ in (8.32) is known. Then each feedback

$$Y_i = A_i, \quad i = 1, \cdots, p,$$

can produce control output B_j, $j = 1, \cdots, m$ as

$$B_1 \ltimes_{\mathscr{B}} \cdots \ltimes_{\mathscr{B}} B_m = M_\Sigma \ltimes_{\mathscr{B}} A_1 \ltimes_{\mathscr{B}} \cdots \ltimes_{\mathscr{B}} A_p. \tag{8.33}$$

8.3.3 Defuzzification

The control output from the controller, equivalent to (8.33), is

$$B := B_1 \ltimes_{\mathscr{B}} \cdots \ltimes_{\mathscr{B}} B_m \in \mathscr{B}^{\infty}_{\beta\times 1}, \tag{8.34}$$

where $\beta = \prod_{i=1}^m \beta_i$.

The purpose of defuzzification is to provide controls $(u_1, \cdots, u_m)$ from fuzzy set B. We first use a simple example to depict it.

Example 8.6. Assume there are two controls u_1 and u_2 with $u_1 \in [-4, 4]$, and $u_2 \in [-6, 6]$. Moveover, their degree-based fuzzy sets are depicted in Fig. 8.3.

Now for u_1 we identify

$$V_1 := NB \sim \delta_5^5;\ V_2 := NS \sim \delta_5^4;\ V_3 := ZO \sim \delta_5^3;\ V_4 := PS \sim \delta_5^2;\ V_5 := PB \sim \delta_5^1.$$

For u_2 we identify

$$\begin{array}{l} W_1 = NB \sim \delta_7^7;\ W_2 := NM \sim \delta_7^6;\ W_3 := NS \sim \delta_7^5;\ W_4 := ZO \sim \delta_7^4; \\ W_5 := PS \sim \delta_7^3;\ W_6 := PM \sim \delta_7^2;\ W_7 := PB \sim \delta_7^1 \quad . \end{array}$$

Then we have

$$\mu_{V_i\times W_j}(u_1, u_2) = \mu_{V_i}(u_1) \wedge \mu_{W_j}(u_2), \quad i = 1, \cdots, 5;\ j = 1, \cdots, 7. \tag{8.35}$$

Now we consider

$$\mu_{V_i\times W_j}(u_1, u_2) = \mu_{V_i}(u_1) \wedge \mu_{W_j}(u_2), \quad i = 1, \cdots, 5;\ j = 1, \cdots, 7. \tag{8.36}$$

Note that in this example we have unique

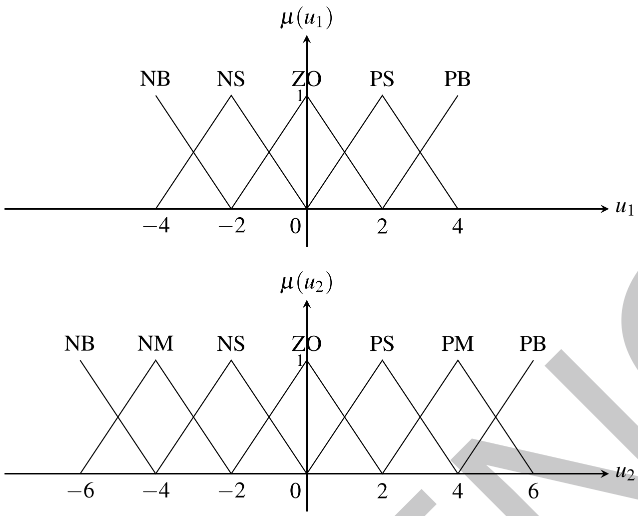

Fig. 8.3 Degree-based fuzzy sets of u_1 and u_2

$$\mu^{-1}_{V_i\times W_j}(1), \quad i=1,\cdots,5;\ j=1,\cdots,7. \tag{8.37}$$

For instance,

$$\mu^{-1}_{V_1\times W_1}(1)=(4,6); \quad \mu^{-1}_{V_1\times W_2}(1)=(4,4),\cdots$$

Then we may choose $\mu^{-1}_{V_i\times W_j}(1)$ as the defuzzificated value of $\delta_5^i \ltimes \delta_7^j$.

Denote the fuzzy values obtained from (8.34) as

$$B=[b_{11}\ \cdots\ b_{17}\ b_{21}\ \cdots\ b_{27}\ \cdots\ b_{51}\ \cdots\ b_{57}]^T.$$

Then we take the weighted values as the defuzzificated controls. That is,

$$(u_1,u_2)=\sum_{i=1}^{5}\sum_{j=1}^{7}\left(\frac{b_{i,j}}{\sum\limits_{i=1}^{5}\sum\limits_{j=1}^{7}b_{ij}}\right)\mu^{-1}_{V_i\times W_j}(1). \tag{8.38}$$

We can extend the procedure proposed in Example 8.6 to general case. Assume the degree-based fuzzy sets for controls are of the forms as isosceles triangle or isosceles trapezoid. Then

$$\mu^{-1}_{U^1_{i_1}\times\cdots U^m_{im}}(1), \quad i_s=1,\cdots,\beta_s;\ s=1,\cdots,m$$

are either a point or a segment. Then we use $\overline{\mu^{-1}_{U^1_{i_1}\times\cdots U^m_{im}}(1)}$ both the point value or the average value of the segment.

Next, assume the fuzzy values obtained from (8.34) are

$$B = [b_{1\cdots 11} \ \cdots b_{1\cdots 1\beta_m} \cdots b_{1\beta_2\cdots\beta_m} \ \cdots b_{\beta_1\beta_2\cdots\beta_m}].$$

Then the defuzzificated controls are chosen as

$$(u_1,\cdots,u_m) = \sum_{j_1=1}^{\beta_1}\cdots\sum_{j_m=1}^{\beta_m}\left(\frac{b_{j_1\cdots j_m}}{\sum\limits_{i_1=1}^{\beta_1}\cdots\sum\limits_{i_m=1}^{\beta_m} b_{i_1\cdots i_m}}\right)\overline{\mu^{-1}_{U^1_{j_1}\times\cdots U^m_{j_m}}(1)}. \tag{8.39}$$

Exercise 8

1. (to be completed).

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