Chapter 4
Right and General Semi-tensor Products

In the previous two chapters the left STP has been discussed. A natural question is: can we define the right STP? If yes, what is its relationship with left STP? Secondly, the left STP has only been defined over two factor matrices satisfying the multiplier dimension case. Another natural question is: Can we define the STP for two arbitrary matrices? This chapter is devoted to the right STP and both left and right STPs on arbitrary matrices.

4.1 Right STP

First, we recall the Kronecker product of matrices. Recall the Proposition 1.4, which says that assume $A \in \mathscr{M}_{m\times n}$ and $B \in \mathscr{M}_{p\times q}$, then

$$A \otimes B = (A \otimes I_p)(I_n \otimes B). \tag{4.1}$$

According to Proposition 2.4, the left semi-tensor product has an alternative definition as

$$A \ltimes B = \begin{cases} (A \otimes I_t)B, & A \prec_t B, \\ A(B \otimes I_t), & A \succ_t B. \end{cases} \tag{4.2}$$

Compared to (4.1), (4.2) seems to be obtained by "only making a left identity matrix matching". This is also a reason for calling it the semi-tensor product.

Now assume two matrices satisfy the requirement of multiplier dimension, then it is very natural that we can also make a right identity matrix matching. Hence the following definition is very natural.

Definition 4.1. Given two matrices A and B. Assume either $A \prec_t B$ or $A \succ_t B$. Then the right STP of A and B, denoted by $A \rtimes B$, is defined as

$$A \rtimes B = \begin{cases} (I_t \otimes A)B, & A \prec_t B, \\ A(I_t \otimes B), & A \succ_t B. \end{cases} \tag{4.3}$$

Many properties of right semi-tensor product are paralleled to its left counterpart. We state the following properties, and leave the proves to the reader.

Proposition 4.1. *The right STP has the following properties.*

1. *(Associative Law):*

$$(A \rtimes B) \rtimes C = A \rtimes (B \rtimes C), \tag{4.4}$$

2. *(Distributive Law):*

$$(A + B) \rtimes C = A \rtimes C + B \rtimes C, \quad C \rtimes (A + B) = C \rtimes A + C \rtimes B. \tag{4.5}$$

3. *Let X and Y be two column vectors. Then*

$$X \rtimes Y = Y \otimes X. \tag{4.6}$$

Let X and Y be two row vectors. Then

$$X \rtimes Y = X \otimes Y. \tag{4.7}$$

4. *Assume $A \rtimes B$ is defined. Then*

$$(A \rtimes B)^{\mathrm{T}} = B^{\mathrm{T}} \rtimes A^{\mathrm{T}}. \tag{4.8}$$

5. *If $M \in \mathscr{M}_{m \times pn}$, then*

$$M \rtimes I_n = M; \tag{4.9}$$

If $M \in \mathscr{M}_{m \times n}$, then

$$M \rtimes I_{pn} = I_p \otimes M; \tag{4.10}$$

If $M \in \mathscr{M}_{pm \times n}$, then

$$I_p \rtimes M = M; \tag{4.11}$$

If $M \in M_{m \times n}$, then

$$I_{pm} \rtimes M = I_p \otimes M. \tag{4.12}$$

Proposition 4.2. *Assume A and B are two square matrices with proper dimensions such that $A \rtimes B$ is well defined. Then*

1. *$A \rtimes B$ and $B \rtimes A$ have the same characteristic functions.*

2.

$$\operatorname{tr}(A \rtimes B) = \operatorname{tr}(B \rtimes A). \tag{4.13}$$

3. If both A and B are orthogonal matrices (upper triangular matrices, lower triangular matrices, or diagonal matrices), then so is $A \rtimes B$.
4. If either A or B is (or both are) invertible , then $A \rtimes B \sim B \rtimes A$.
5. If both A and B are invertible, then

$$(A \rtimes B)^{-1} = B^{-1} \rtimes A^{-1}. \tag{4.14}$$

6. If $A \prec_t B$, then

$$\det(A \rtimes B) = [\det(A)]^t \det(B); \tag{4.15}$$

If $A \succ_t B$, then

$$\det(A \rtimes B) = \det(A)[\det(B)]^t. \tag{4.16}$$

Recall that in Chapter 2 we define the left semi-tensor product in two steps: First, we define the left STP of two vectors in Definition 2.2, and then use it to define the STP to two matrices and in the Definition 2.3. Can the right STP also been defined in this way? We first consider the vector case.

Definition 4.2. Let $X = (x_1, \cdots, x_s)$ be a row vector and $Y = (y_1, \cdots, y_t)^{\mathrm{T}}$ be a column vector.

Case 1: If $s = t \times n$, where $n \in \mathbb{N}$, then we define

$$X \rtimes Y := \left(X^1 Y \; X^2 Y \; \cdots \; X^t Y\right) \in \mathbb{R}^n, \tag{4.17}$$

where $X = (X^1 \; X^2 \; \cdots \; X^t)$, $X^i \in \mathbb{R}^n$, $i = 1, \cdots, t$.
Case 2: If $t = s \times n$, where $n \in \mathbb{N}$, then we define

$$X \rtimes Y := \begin{bmatrix} XY^1 \\ XY^2 \\ \vdots \\ XY^n \end{bmatrix} \in \mathbb{R}^n. \tag{4.18}$$

It is easy to verify that when both X and Y are vectors, Definition 4.2 coincides with Definition 4.1. Unfortunately, Definition 4.2 can not be extended to the right STP of matrices.

In fact, the row-column multiplication rule is a particular case of the block multiplication rule, as demonstrated in Proposition 2.2. The following example shows that the right STP does not satisfy the block multiplication rule.

Example 4.1. Let $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2)^{\mathrm{T}}$. Then by definition

$$A \rtimes B = (a_1 b_1 + a_2 b_2 \ , \ a_3 b_1 + a_4 b_2). \tag{4.19}$$

If we split A and B into blocks as $A = (A_1 \ \ A_2)$ and $B = (B_1 \ \ B_2)^{\mathrm{T}}$, then according to the block multiplication rule, we have

$$A_1 \rtimes B_1 + A_2 \rtimes B_2 = (a_1 \ \ a_2) \rtimes (b_1) + (a_3 \ \ a_4) \rtimes (b_2) = (a_1 b_1 + a_3 b_2 \ \ a_2 b_1 + a_4 b_2). \tag{4.20}$$

(4.20) differs from (4.19), which means (4.20) is incorrect.

Though both left STP and right STP are the generalization of conventional matrix product, but in most cases the left STP is more useful than the right STP. One of the reasons is that the right STP does not satisfy block multiplication rule. Another basic reason is the left STP has very clear physical meaning in representing multi-dimensional data. We, therefore, consider the left STP as default STP.

The following proposition shows how to convert one STP to the other.

Proposition 4.3. *Given $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. If $A \succ_t B$, then*

$$A \rtimes B = A \ltimes W_{[p,t]} \ltimes B \ltimes W_{[t,q]}. \tag{4.21}$$

Conversely,

$$A \ltimes B = A \rtimes W_{[t,p]} \rtimes B \rtimes W_{[q,t]}. \tag{4.22}$$

If $A \prec_t B$, then

$$A \rtimes B = W_{[m,t]} \ltimes A \ltimes W_{[t,n]} \ltimes B. \tag{4.23}$$

Conversely,

$$A \ltimes B = W_{[t,m]} \rtimes A \rtimes W_{[n,t]} \rtimes B. \tag{4.24}$$

Proof. We prove (4.21) only. The proves of (4.22)$-$(4.24) are similar. Using (3.47), we have

$$\begin{aligned} A \rtimes B &= A(I_t \otimes B) = AW_{[p,t]}(B \otimes I_t)W_{[t,q]} \\ &= A \ltimes W_{[p,t]} \ltimes B \ltimes W_{[t,q]}. \end{aligned}$$

□

Particularly, when the STP of vectors are considered we have the following corollary.

Corollary 4.1. *1. Let X be a row vector of dimension np and Y a column vector of dimension p. Then*

$$X \rtimes Y = \left(XW_{[p,n]}\right) \ltimes Y. \tag{4.25}$$

Conversely,

$$X \ltimes Y = \left(XW_{[n,p]}\right) \rtimes Y. \tag{4.26}$$

2. *Let a row* $\dim(X) = p$ *and a column* $\dim(Y) = pn$. *Then*

$$X \rtimes Y = X \ltimes \left(W_{[n,p]}Y\right). \tag{4.27}$$

Conversely,

$$X \ltimes Y = X \ltimes \left(W_{[p,n]}Y\right). \tag{4.28}$$

Proof. According to 4.3, (4.25)–(4.28) are the particular cases of (4.21)–(4.24) respectively. □

We can also reveal the physical meaning of right STP through the multi-dimensional mappings. The right STP can also search the hierarchy structure of data, or find the "pointer", "pointer to pointer" etc. Let $\sigma : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{s} \to \mathbb{R}^s$ be a multilinear mapping. For each basis of $\mathbb{R}^n$ denoted by $\{e_1, \cdots, e_n\}$ and a basis of $\mathbb{R}^s$ denoted by $\{d_1, \cdots, d_s\}$, assume

$$\sigma(e_{i_1}, e_{i_2}, \cdots, e_{i_s}) = \sum_{k=1}^{s} \alpha^k_{i_1,\cdots,i_s} d_k.$$

then the structure matrix of M_σ is defined as

$$M_\sigma = \begin{pmatrix} \alpha^1_{1\cdots 11} & \alpha^1_{1\cdots 12} & \cdots & \alpha^1_{n\cdots nn} \\ \alpha^2_{1\cdots 11} & \alpha^2_{1\cdots 12} & \cdots & \alpha^2_{n\cdots nn} \\ \vdots & \vdots & & \vdots \\ \alpha^n_{1\cdots 11} & \alpha^n_{1\cdots 12} & \cdots & \alpha^n_{n\cdots nn} \end{pmatrix}. \tag{4.29}$$

For $X_1, \cdots, X_s \in \mathbb{R}^n$, express each X_i by its coefficient vector, that is, if $X_i = \sum_{j=1}^n x^i_j e_j$, then its vector form is $X_i = (x^i_1, \cdots, x^i_n)^{\mathrm{T}}$. Then we have

$$\sigma(X_1, \cdots, X_s) = M_\sigma \ltimes X_1 \ltimes X_2 \ltimes \cdots \ltimes X_s = M_\sigma \rtimes X_k \rtimes X_{k-1} \ltimes \cdots \rtimes X_1. \tag{4.30}$$

In applications, the right STP is sometimes convenient. Corresponding to (2.49) and (2.50), we have the following conclusions.

Proposition 4.4. *Assume* $A \in \mathcal{M}_{m\times n}$, $X \in \mathcal{M}_{n\times q}$, $Y \in \mathcal{M}_{p\times m}$, *then the stack forms of the products are*

$$V_r(YA) = (I_p \otimes A^{\mathrm{T}})V_r(Y) = A^{\mathrm{T}} \rtimes V_r(Y); \tag{4.31}$$
$$V_c(AX) = (I_q \otimes A)V_c(X) = A \rtimes V_c(X); \tag{4.32}$$

Proof. To prove (4.31), we use Table 3.1 to get that

$$\begin{aligned} V_r(YA) &= V_c(A^{\mathrm{T}}Y^{\mathrm{T}}) \\ &= \left(I_p \otimes A^{\mathrm{T}}\right) V_c(Y^{\mathrm{T}}) \left(I_p \otimes A^{\mathrm{T}}\right) V_r(Y) \\ &= A^{\mathrm{T}} \rtimes V_r(Y). \end{aligned}$$

As for (4.32), Using Table 3.3 and the equality (2.51), we have

$$\begin{aligned} V_c(AX) &= W_{[m,q]} V_r(AX) = W_{[m,q]} \ltimes A \ltimes V_r(X) \\ &= W_{[m,q]} \ltimes A \ltimes W_{[q,p]} \ltimes V_c(X) \\ &= (I_q \otimes A) \ltimes V_c(B) = A \rtimes V_c(B). \end{aligned}$$

□

Using Proposition 4.4, we can deduce some formulas for the product of three matrices.

Proposition 4.5.

$$V_r(ABC) = (A \otimes C^{\mathrm{T}}) V_r(B); \tag{4.33}$$

$$V_c(ABC) = (C^{\mathrm{T}} \otimes A) V_c(B). \tag{4.34}$$

Proof. Using (2.49) and (4.30), we have

$$\begin{aligned} V_r(ABC) &= (A \otimes I_q) V_r(BC) = (A \otimes I_q)(I_n \otimes C^{\mathrm{T}}) V_r(B) \\ &= (A \otimes C^{\mathrm{T}}) V_r(B). \end{aligned}$$

This proves (4.33).

Using (2.50) and (4.30), we have

$$\begin{aligned} V_c(ABC) &= (I_q \otimes A) V_c(BC) = (I_q \otimes A)(C^{\mathrm{T}} \otimes I_n) V_c(B) \\ &= (C^{\mathrm{T}} \otimes A) V_c(B). \end{aligned}$$

This proves (4.34). □

The following proposition is an immediate consequence of the definition.

Proposition 4.6. *Assume $A \in \mathscr{M}_{m \times n}$ and $B \in \mathscr{M}_{n \times m}$. Then*

$$\mathrm{tr}(AB) = V_c^{\mathrm{T}}(A) V_r(B) = V_r^{\mathrm{T}}(B) V_c(A) = V_c^{\mathrm{T}}(B) V_r(A) = V_r^{\mathrm{T}}(A) V_c(B). \tag{4.35}$$

Combining Proposition 4.6 with (4.33) and (4.34) yields

Proposition 4.7. *Let A, B, C, D are of proper dimensions such that the conventional matrix product of $ABCD$ is well defined and is a square matrix. Then*

$$\mathrm{tr}(ABCD) = V_c^{\mathrm{T}}(A)(B \otimes D^{\mathrm{T}}) V_r(C) = V_r^{\mathrm{T}}(C)(B^{\mathrm{T}} \otimes D) V_c(A). \tag{4.36}$$

Finally, we give two propositions, which are convenient when we convert a matrix polynomial into a polynomial of its entries.

First we define the power of right semi-tensor product: Let $A \in \mathscr{M}_{m\times n}$, and $n|m$ or $m|n$. Then $A \rtimes A$ is well defined, and hence the power of right-tensor product can also be defined as

$$A^{\rtimes k} = \underbrace{A \rtimes A \rtimes \cdots \rtimes A}_{k}.$$

Note that as a convention, we define

$$A^{k} = \underbrace{A \ltimes A \ltimes \cdots \ltimes A}_{k}.$$

Hence for the power of right-tensor product the symbol $\rtimes$ on the power should not be omitted. It is obvious that when A is a square matrix the powers of both the left and the right semi-tensor products are the same.

Now we are ready to state the propositions.

Proposition 4.8. *Let $Z \in \mathscr{M}_n$. Then*

$$Z^k = (F^{\rtimes k}) \ltimes V_c^k(Z), \quad k \geq 1, \tag{4.37}$$

where

$$F = \left(I_n \otimes V_c^{\mathrm{T}}(I_n)\right) \ltimes W_{[n]}. \tag{4.38}$$

Proof. We prove (4.37) by mathematical induction. Using (3.56), we have

$$Z = \pi_n^c(I_n) \ltimes V_c(Z).$$

Let $F = \pi_n^c(I_n)$. Using (6.47) and (3.52), it is easy to see that F has the form as in (4.38). Note that $F \in \mathscr{M}_{n\times n^3}$, it follows that $F^{\rtimes k} \in M_{n\times n^{2k+1}}$. Next, we assume (4.37) holds for k. Then using (2.55), we have

$$\begin{aligned}
Z^{k+1} &= Z \ltimes Z^k = F \ltimes V_c(Z) \ltimes F^{\rtimes k} \ltimes V_c^k(Z) \\
&= F \ltimes (I_{n^2} \otimes F^{\rtimes k}) \ltimes V_c(Z) \ltimes V_c^k(Z) \\
&= \left(F(I_{n^2} \otimes F^{\rtimes k})\right) \ltimes V_c^{k+1}(Z) \\
&= F^{\rtimes (k+1)} \ltimes V_c^{k+1}(Z).
\end{aligned}$$

□

Proposition 4.9. *Let $Z \in \mathscr{M}_n$. Then*

$$V_r(Z^k) = \left(E^{\rtimes (k-1)}\right) V_r^k(Z), \quad k \geq 2, \tag{4.39}$$

$$V_c(Z^k) = W_{[n]} \left(E^{\rtimes (k-1)}\right) (W_{[n]})^{\otimes k} V_c^k(Z), \quad k \geq 2, \tag{4.40}$$

where

$$E = I_n \otimes V_c^{\mathrm{T}}(I_n).$$

Proof. First, we prove (4.39). For $k = 1$, we have

$$V_r(Z^2) = Z \ltimes V_r(Z) = FV_c(Z)V_r(Z) = FW_{[n]}V_r^2(Z) = EV_r^2(Z).$$

Now we assume (4.39) is true for k, Then from (2.38) we have

$$\begin{aligned} V_r(Z^{k+1}) &= Z \ltimes V_r(Z^k) = E \ltimes V_r(Z) \ltimes E^{\rtimes(k-1)} \ltimes V_r^k(Z) \\ &= \left(E \ltimes (I_{n^2} \otimes E^{\rtimes(k-1)})\right) \ltimes V_r(Z) \ltimes V_r^k(Z) \\ &= \left(E^{\rtimes k}\right) \ltimes V_r^{k+1}(Z). \end{aligned}$$

This proves (4.39).

Note that

$$V_r^k(Z) = (\underbrace{W_{[n]} \otimes \cdots \otimes W_{[n]}}_{k})V_c^k(Z) = (W_{[n]})^{\otimes k}V_c^k(Z).$$

Left multiplying $W_{[n]}$ to both sides of (4.39) yields (4.40). □

4.2 Semi-tensor Product of Arbitrary Matrices

This section considers the left and right semi-tensor products for two arbitrary matrices. For statement ease, we call them the general semi-tensor product of matrices.

Let $a, b \in \mathbb{Z}^+$. Denote the least common multiplier of a and b by $\mathrm{lcm}\{a, b\}$.

Definition 4.3. Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$ and $\alpha = \mathrm{lcm}\{n, p\}$.

1. The general left STP of A and B is defined as as

$$A \ltimes B = \left(A \otimes I_{\frac{\alpha}{n}}\right)\left(B \otimes I_{\frac{\alpha}{p}}\right). \tag{4.41}$$

2. The general right STP of A and B is defined as

$$A \rtimes B = \left(I_{\frac{\alpha}{n}} \otimes A\right)\left(I_{\frac{\alpha}{p}} \otimes B\right). \tag{4.42}$$

Note that when $n = p$ the general left (right) STP of matrices becomes the conventional matrix product.

If $n \vee p = n$ or $n \vee p = p$, then the general left (right) STP of matrices becomes the previously defined left (right) STP. Unless otherwise stated, through this book the STP is defined for multiplier dimensional case except this section.

The reason that we did not pay much attention to the general STP is that unlike the multiplier dimensional case, we could not find the reasonable physical explanation for the general STP and did not find meaningful applications so far.

In the following we consider some basic properties of general left (right) STP.

Proposition 4.10. *The general left (right) STP satisfies*

1. (Distributive Law)

$$(A+B)\ltimes C=(A\ltimes C)+(B\ltimes C), \tag{4.43}$$
$$(A+B)\rtimes C=(A\rtimes C)+(B\rtimes C), \tag{4.44}$$
$$C\ltimes (A+B)=(C\ltimes A)+(C\ltimes B), \tag{4.45}$$
$$C\rtimes (A+B)=(C\rtimes A)+(C\rtimes B). \tag{4.46}$$

2. (Associative Law)

$$(A\ltimes B)\ltimes C=A\ltimes (B\ltimes C), \tag{4.47}$$
$$(A\rtimes B)\rtimes C=A\rtimes (B\rtimes C). \tag{4.48}$$

Proof. Distributive law is easily verifiable. We prove the associative law.

Let $A\in\mathscr{M}_{m\times n}$, $B\in\mathscr{M}_{p\times q}$, $C\in M_{s\times t}$. We first show that both sides of (4.47)–(4.48) can be expressed as

$$(A\otimes I_\alpha)(A\otimes I_\beta)(A\otimes I_\gamma). \tag{4.49}$$

This equality is obtained from the equality that

$$(A\otimes I_m)(B\otimes I_n)\otimes I_s=(A\otimes I_{m+s})(B\otimes I_{n+s}).$$

From the definition of the general semi-tensor product one sees that no matter what is the order the product is executed, we finally need to find the smallest natural numbers α, β, γ, such that

$$\begin{cases}\alpha n=\beta p\\ \beta q=\gamma s.\end{cases} \tag{4.50}$$

If we can show that the smallest solution of (4.50) is unique, the conclusion follows. Since $\alpha nq=\gamma sp$, we can assume $\mathrm{lcm}\{nq,sp\}=h$. Then

$$\alpha=\mu\frac{sp}{h},\quad \gamma=\mu\frac{nq}{h}.$$

It follows that

$$\beta=\frac{\mu sn}{h}=\frac{sn}{h/\mu}.$$

Now the smallest μ satisfying $c=(h/\mu)$ is the least common multiplier of nq, sp and sn. Define

$$c=nq\wedge sp\wedge sn,$$

then we have that the both sides of (4.47) and (4.48) equal to (4.44). Moreover,

$$\alpha = \frac{sp}{c}, \quad \beta = \frac{sn}{c}, \quad \gamma = \frac{nq}{c}.$$

□

Almost all the major properties of the conventional matrix product remain true for generalized left (right) semi-tensor product of matrices. We list the major properties as follows:

Proposition 4.11. *1.*

$$\begin{cases} (A \ltimes B)^{\mathrm{T}} = B^{\mathrm{T}} \ltimes A^{\mathrm{T}}, \\ (A \rtimes B)^{\mathrm{T}} = B^{\mathrm{T}} \rtimes A^{\mathrm{T}}. \end{cases} \tag{4.51}$$

2. If $M \in \mathscr{M}_{m\times pn}$, then

$$\begin{cases} M \ltimes I_n = M, \\ M \rtimes I_n = M; \end{cases} \tag{4.52}$$

If $M \in \mathscr{M}_{pm\times n}$, then

$$\begin{cases} I_m \ltimes M = M, \\ I_m \rtimes M = M. \end{cases} \tag{4.53}$$

In the following items, A and B are two square matrices.

3. $A \ltimes B$ and $B \ltimes A$ ($A \rtimes B$ and $B \rtimes A$) have the same characteristic function.

4.

$$\begin{cases} \mathrm{tr}(A \ltimes B) = \mathrm{tr}(B \ltimes A), \\ \mathrm{tr}(A \rtimes B) = \mathrm{tr}(B \rtimes A). \end{cases} \tag{4.54}$$

5. If both A and B are orthogonal matrices (upper triangular matrices, down triangular matrices, or diagonal matrices), then so is $A \ltimes B$ ($A \rtimes B$).

6. If at least one of A and B is invertible, then $A \ltimes B \sim B \ltimes A$ ($A \rtimes B \sim B \rtimes A$).

7. If both A and B are invertible, then

$$\begin{cases} (A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}, \\ (A \rtimes B)^{-1} = B^{-1} \rtimes A^{-1}. \end{cases} \tag{4.55}$$

8. The determinant of the product satisfies

$$\begin{cases} \det(A \ltimes B) = [\det(A)]^{\frac{\alpha}{n}} [\det(B)]^{\frac{\alpha}{p}}, \\ \det(A \rtimes B) = [\det(A)]^{\frac{\alpha}{n}} [\det(B)]^{\frac{\alpha}{p}}, \end{cases} \tag{4.56}$$

where $\alpha = \mathrm{lcm}\{n, p\}$.

The following property is for general left STP only.

Proposition 4.12. *Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Then*

$$C = A \ltimes B = (C^{ij}), \quad i = 1, \cdots, m, \ j = 1, \cdots, q, \tag{4.57}$$

where

$$C^{ij} = A^i \ltimes B_j.$$

Here $A^i = \mathrm{Row}_i(A)$ and $B_j = \mathrm{Col}_j(B)$.

Note that (4.57) can also be considered as the definition of general left STP. In equal dimensional case, C^{ij} is a number, in multiplier case it is a vector, and in general case it is a block.

Remark 4.1. As a convention, for any two matrices $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$, the default matrix product is assumed to be the left STP. That is,

$$AB = A \ltimes B. \tag{4.58}$$

when $n = p$, it is the conventional matrix product; when $n \vee p = \max\{n, p\}$, it is the left STP defined in Chapter 2; and otherwise, it is the general left STP. Under this convention, the symbol $\ltimes$ is usually omitted, unless we would like to emphasize it is STP.

The reader may be convinced by the discussion so far that the concept of conventional matrix product can be replaced by the left STP.

Let $A \in \mathcal{M}_{m\times n}$. We can define the power of general left STP of A as

$$\begin{cases} A^1 = A, \\ A^{k+1} = A \ltimes A^k, \quad k \geq 1. \end{cases}$$

Similarly, we can also define the power of general left STP of A as

$$\begin{cases} A^{\rtimes 1} = A, \\ A^{\rtimes (k+1)} = A \rtimes A^{\rtimes k}, \quad k \geq 1. \end{cases}$$

To consider the dimension of A^k (or $A^{\rtimes k}$), let $\mathrm{lcm}\{m, n\} = t$. Set $m = m_0 t$ and $n = n_0 t$, then m_0 and n_0 are co-prime. It is easy to prove by mathematical induction that $A^k \in \mathcal{M}_{m_0^k t \times n_0^k t}$ ($A^{\rtimes k} \in \mathcal{M}_{m_0^k t \times n_0^k t}$).

Exercise 4

1. Prove (4.6) and (4.7).
2. Prove (4.8).
3. Prove (4.10)–(4.12).

4. In cubic matrix theory, let
$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_m \end{bmatrix},$$
where X_i, $i = 1, \cdots, n$ are $n \times n$ matrices. Assume $b \in \mathbb{R}^n$, the quadratic form of X is defined as [1]
$$b^{\mathrm{T}} X b := \begin{bmatrix} b^{\mathrm{T}} X_1 b \\ \vdots \\ b^{\mathrm{T}} X_m b \end{bmatrix}.$$
Prove that it can be expressed as
$$b^{\mathrm{T}} X b = b^{\mathrm{T}} \rtimes (Xb) = (b^{\mathrm{T}} \rtimes X) b.$$
5. Give the matrix-vector expression of $V_r(AX)$ using the right semi-tensor product.
6. Consider
$$GL(\mathbb{R}) := \cup_{n=1}^{\infty} GL(n, \mathbb{R}).$$
Define an equivalence on $GL(\mathbb{R})$ as
$$A \sim A \otimes I_k, \quad k \in \mathbb{N}.$$
Denote the equivalent class of A by $[A]$. Consider the quotient set
$$G := GL(\mathbb{R}) / \sim = \{[A] | A \in GL(\mathbb{R})\}.$$
Define the product over G as
$$[A] \ltimes [B] := [A \ltimes B].$$
   a. Prove that $(G, \ltimes)$ is a group.
   b. Let
   $$O(\mathbb{R}) := \cup_{n=1}^{\infty} O(n, \mathbb{R}).$$
   Define $G_O := O(\mathbb{R}) / \sim$. Prove that G_O is a subgroup of G.

References

1. Wang, X.: Parameter Estimate of Nonlinear Models — Theory and Applications. Wuhan Univ. Press, Wuhan (2002). (in Chinese)