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Optimal Birth Control of Population Dynamics. II. Problems with Free Final Time, Phase Constraints, and Mini-Max Costs*

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We study optimal birth control of population systems of McKendrick type which is a distributed parameter system involving first order partial differential equations with nonlocal bilinear boundary control. New results on problems with free final time, phase constraints, and mini-max costs are presented.

In [1] we discussed optimal birth control of population systems of McKendrick type. The present article (which is a direct continuation of [1]) presents further new results of current interests. These include problems with free final time, of which the minimum time problem is a special case (but relaxing many convexity assumptions). Systems with phase constraints are also studied. Finally, mini-max control for population regulation is characterized. It is assumed that the reader is familiar with the terminology and notation of [1].

5. FREE FINAL TIME PROBLEM

Consider the free final time optimal control problem of the population control system

$$\text{Problem (P): Minimize } J(\beta, p) = \int_0^{t_1} \int_0^{r_m} L(p(r, t), \beta(t))\, dr\, dt$$

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subject to

$$\frac{\partial p(r,t)}{\partial t}+\frac{\partial p(r,t)}{\partial r}=-\mu(r)\,p(r,t),\qquad 0<r<r_m,\quad t>0,$$

$$p(r,0)=p_0(r),\qquad 0\leqslant r\leqslant r_m,$$

$$p(0,t)=\beta(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p(r,t)\,dr,\qquad t\geqslant 0,$$

$$p(r,t_1)=p^0(r),\qquad t_1>0,\ \beta(t)\in M\subset\mathbb{R}^+, \tag{1}$$

where L is a function defined on $L^2(0, r_m)\times\mathbb{R}^+$ satisfying

(1) $L(p(r), \beta)$ is continuous in β,

(2) $|\partial L(p(r), \beta)/\partial p|$ is bounded for every bounded subset of $L^2(0, r_m)\times\mathbb{R}^+$.

For any measurable function $v(s)\geqslant 0$, define the time transformation

$$t(\tau)=\int_0^{\tau} v(s)\,ds,\qquad t(1)=t_1 \tag{2}$$

and let $p(r, \tau)=p(r, t(\tau))$,

$$\beta(\tau)=\begin{cases}\beta(t(\tau)), & \tau\in S_1,\\ \text{arbitrary}, & \tau\in S_2,\end{cases} \tag{3}$$

then $(p(r,\tau), \beta(\tau))$ satisfies

$$\frac{\partial p(r,\tau)}{\partial \tau}+v(\tau)\frac{\partial p(r,\tau)}{\partial r}=-\mu(r)\,v(\tau)\,p(r,\tau),\qquad 0<r<r_m,\quad 0\leqslant\tau\leqslant 1,$$

$$p(r,0)=p_0(r),\qquad 0\leqslant r\leqslant r_m,$$

$$v(\tau)\,p(0,\tau)=v(\tau)\,\beta(\tau)\int_{r_1}^{r_2} k(r)\,h(r)\,p(r,\tau)\,dr,\qquad 0\leqslant\tau\leqslant 1,$$

$$p(r,1)=p^0(r), \tag{4}$$

where

$$S_1=\{\tau\mid \tau\in[0,1],\ v(\tau)>0\},$$
$$S_2=\{\tau\mid \tau\in[0,1],\ v(\tau)=0\}. \tag{5}$$

Conversely, if $(p(r,\tau), \beta(\tau))$ solves Eq. (4), define $p(r,t)=p(r,\tau(t))$, $\beta(t)=\beta(\tau(t))$,

$$\tau(t)=\inf\{\tau\mid t(\tau)=t\} \tag{6}$$

then $(p(r, t), \beta(t))$ satisfies Eq. (1) for $t = t(\tau)$, $v(\tau) > 0$, but for the monotone function $t(\tau) = \int_0^\tau v(s)\,ds$, $mes\{t = t(\tau) \mid v(\tau) > 0\} = t_1 = \int_0^1 v(s)\,ds$, so $(p(r, t), \beta(t))$ satisfies Eq. (1) for $t \in [0, t_1]$ a.e.

Based on the above arguments, we consider the optimal (fixed final time) problem

$$\text{Problem (Q): Minimize } J(\beta, p) = \int_0^{t_1} \int_0^{r_m} L(p(r, t), \beta(t))\,dr\,dt \text{ subject to Eq. (4).}$$

If (p^*, β^*, t_1) solves Problem (P), then for any $v^*(\tau) \geqslant 0$ satisfying $\int_0^1 v^*(s)\,ds = t_1$, $\beta^*(\tau)$ defined similar to (3), $(p^*(r, \tau), \beta^*(\tau), v^*)$ solves Problem (Q). By this $\beta(\tau)$, we put forward another problem as

$$\text{Problem (L): Minimize } J(\beta^*, p, v) = \int_0^1 \int_0^{r_m} v(\tau)\, L(p(r, \tau), \beta^*(\tau))\,dr\,d\tau$$

subject to

$$\frac{\partial p(r, \tau)}{\partial \tau} + v(\tau) \frac{\partial p(r, \tau)}{\partial r} = -\mu(r)\, v(\tau)\, p(r, \tau), \qquad 0 < r < r_m, 0 \leqslant \tau \leqslant 1,$$

$$p(r, 0) = p_0(r), \qquad 0 \leqslant r \leqslant r_m,$$

$$v(\tau)\, p(0, \tau) = v(\tau)\, \beta^*(\tau) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, \tau)\,dr, \qquad 0 \leqslant \tau \leqslant 1,$$

$$p(r, 1) = p^0(r), \tag{7}$$

and (p^*, v^*) solves Problem (L). Consider the solution of (7) as that of the integral equation

$$\int_0^r p(s, \tau)\,ds - \int_0^r p_0(s)\,ds + \int_0^\tau v(\xi) \left[p(r, \xi) - \beta^*(\xi) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, \xi)\,dr \right] d\xi$$
$$+ \int_0^r \int_0^\tau v(\xi)\, \mu(s)\, p(s, \xi)\,ds\,d\xi \tag{8}$$

$$p(r, 1) = p^0(r). \tag{9}$$

Similarly, we consider that the solution of the differential equation is equivalent to that of the corresponding integral equation.

Simple arguments can be found that the Eq. (8) has a unique solution on $C(0, 1; L^2(0, r_m))$ and so we take $X = C(0, 1; L^2(0, r_m)) \times L^\infty(0, 1)$ as the state space. Define the inequality constraint

$$\Omega_1 = \{(p(r, \tau), v(\tau)) \in X \mid v(\tau) \geqslant 0, \text{ for } \tau \in [0, 1] \text{ a.e.}\} \tag{10}$$

and the equality constraint

$$\Omega_2=\{(p(r,\tau), v(\tau))\in X \mid (p,v) \text{ satisfies } (8) \text{ and} (9)\}. \tag{11}$$

Under these notations, we can write problem (L) as

$$\begin{aligned}&\text{Minimize } J(\beta^*, p, v)=\int_0^1\int_0^{r_m} v(\tau)\, L(p(r,\tau), \beta^*(\tau))\, dr\, d\tau\\ &\text{subject to } (p(r,\tau), v(\tau))\in\Omega_1\cap\Omega_2\subset X.\end{aligned} \tag{12}$$

$J(\beta^*, p, v)$ is Fréchet differentiable at any point $(\hat{p}, \hat{v})$ and

$$J'(\beta^*, p_0, v_0)(p,\beta) =\int_0^1\int_0^{r_m}\left[v_0(\tau)\frac{\partial L(p_0(r,\tau),\beta^*(\tau))}{\partial p}\, p(r,\tau)+v(\tau)\, L(p_0,\beta^*)\right] dr\, d\tau \tag{13}$$

and so the decreasing direction cone of $J'(\beta^*, p_0, v_0)$ at (p^*, v^*) is

$$K_0=\{(p,v) \mid J'(\beta^*, p^*, v^*)(p,v)<0\}. \tag{14}$$

If $K_0\neq\varnothing$, then for any $f_0\in K_0^*$, there exists a constant $\lambda_0\geqslant 0$, such that

$$f_0(p,v) =-\lambda_0\int_0^1\int_0^{r_m} v^*(\tau)\left[\frac{\partial L(p^*(r,\tau),\beta^*(\tau))}{\partial p}\, p(r,\tau)+L(p^*,\beta^*)\, v(\tau)\right] dr\, d\tau. \tag{15}$$

Notice that $\Omega_1=C(0,1;L^2(0,r_m))\times\hat{\Omega}_1$, $\hat{\Omega}_1=\{v(\tau)\in L^\infty(0,1) \mid v(\tau)\geqslant 0\}$ is a closed convex subset of $L^\infty(0,1)$, $\mathring{\Omega}_1=C\times\mathring{\hat{\Omega}}_1\neq\varnothing$, and so the feasible direction cone of Ω_1 at (p^*, v^*) is

$$K_1=\{\lambda(\mathring{\Omega}_1-(p^*,v^*)) \mid \lambda>0\} \tag{16}$$

for any $f_1\in K_1^*$, if $c(t)\in L(0,1)$ such that

$$f_1(p,v)=\int_0^1 c(\tau)\, v(\tau)\, d\tau \tag{17}$$

then [2]

$$c(\tau)[v-v^*(\tau)]\geqslant 0, \qquad \forall v\in(0,\infty),\ \tau\in[0,1] \text{ a.e.} \tag{18}$$

In order to determine the tangent direction cone of Ω_2 at (p^*, v^*), we define the operator as $G: X \to X$

$$G(p, v) = \left[\int_0^r p(s, \tau)\, ds - \int_0^r p_0(s)\, ds + \int_0^\tau v(\xi) \times \left[p(r, \xi) - \beta^*(\xi) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, \xi)\, dr \right] d\xi + \int_0^r \int_0^\tau v(\xi)\, \mu(s)\, p(s, \xi)\, ds\, d\xi,\ p(r, 1) - p^0(r) \right] \tag{19}$$

then

$$\Omega_2 = \{(p, v) \mid G(p, v) = 0\}. \tag{20}$$

Now

$$\begin{aligned} G'(p^*, v^*)(p, v) &= \left[\int_0^r p(s, \tau)\, ds + \int_0^\tau \left[[v(\xi)\, p^*(r, \xi) + v^*(\xi)\, p(r, \xi)] \right.\right. \\ &\quad \left. - \beta^*(\xi) \int_{r_1}^{r_2} k(r)\, h(r) [v(\xi)\, p^*(r, \xi) + v^*(\xi)\, p(r, \xi)]\, dr \right] d\xi \\ &\quad \left. + \int_0^r \int_0^\tau \mu(s) [v(\xi)\, p^*(s, \xi) + v^*(\xi)\, p(s, \xi)]\, ds\, d\xi,\ p(r, 1) \right] \end{aligned} \tag{21}$$

and we solve the equation

$$G'(p^*, v^*)(p, v) = (q, g) \in X;$$

i.e.,

$$\begin{aligned} &\int_0^r p(s, \tau)\, ds + \int_0^\tau \left[[v(\xi)\, p^*(r, \xi) + v^*(\xi)\, p(r, \xi)] \right. \\ &\quad \left. - \beta^*(\xi) \int_{r_1}^{r_2} k(r)\, h(r) [v(\xi)\, p^*(r, \xi) + v^*(\xi)\, p(r, \xi)]\, dr \right] d\xi \\ &\quad + \int_0^r \int_0^\tau \mu(s)\, [v(\xi)\, p^*(s, \xi) + v^*(\xi)\, p(s, \xi)]\, ds\, d\xi = q(r, \tau), \\ &p(r, 1) = g(r). \end{aligned} \tag{22}$$

If the linearized system

$$\frac{\partial p(r,\tau)}{\partial \tau}+v^*(\tau)\frac{\partial p(r,\tau)}{\partial r}$$
$$= -\mu(r)[v(\tau)\,p^*(r,\tau)+v^*(\tau)\,p(r,\tau)]-v(\tau)\frac{\partial p^*(r,\tau)}{\partial r},$$

$$p(r,0)=0,$$

$$v(\tau)\,p^*(0,\tau)+v^*(\tau)\,p(0,\tau)$$
$$=v(\tau)\,\beta^*(\tau)\int_{r_1}^{r_2}k(r)\,h(r)\,p^*(r,\tau)\,dr$$
$$+v^*(\tau)\,\beta^*(\tau)\int_{r_1}^{r_2}k(r)\,h(r)\,p(r,\tau)\,dr \tag{23}$$

is controllable, then let $\hat{p}(r,\tau)=p(r,\tau)+d(r,\tau)$, $d(r,\tau)$ be determined by

$$\int_0^r d(s,\tau)\,ds+\int_0^\tau v^*(\xi)\left[d(r,\xi)-\beta^*(\xi)\int_{r_1}^{r_2}k(r)\,h(r)\,d(r,\xi)\,dr\right]d\xi$$
$$+\int_0^r\int_0^\tau v^*(\xi)\,\mu(s)\,d(s,\xi)\,ds\,d\xi=q(r,\tau),$$

(p,β), $\beta=\hat{\beta}$ solves Eq. (23) and $p(r,1)=g(r)-d(r,1)$, so $(\hat{p},\hat{\beta})$ solves Eq. (22). In this case, the tangent direction cone of Ω_2 at (p^*,v^*) is determined by

$$K_2=\{(p,v)\mid G'(p^*,v^*)(p,v)=0\},$$

i.e.,

$$\frac{\partial p(r,\tau)}{\partial \tau}+v^*(\tau)\frac{\partial p(r,\tau)}{\partial r}$$
$$= -\mu(r)[v(\tau)\,p^*(r,\tau)+v^*(\tau)\,p(r,\tau)]-v(\tau)\frac{\partial p^*(r,\tau)}{\partial r},$$

$$p(r,0)=0,$$

$$v(\tau)\,p^*(0,\tau)+v^*(\tau)\,p(0,\tau)\,p(0,\tau)$$
$$=v(\tau)\,\beta^*(\tau)\int_{r_1}^{r_2}k(r)\,h(r)\,p^*(r,\tau)\,dr+v^*(\tau)\,\beta^*(\tau)\int_{r_1}^{r_2}k(r)\,h(r)\,p(r,\tau)\,dr$$

$$p(r,1)=0. \tag{24}$$

$K_2 = K_{11} \cap K_{12}$, $K_{12} = \{(p, v) \mid p(r, 1) = 0\}$, K_{11} consists of such $(p, v) \in X$ such that

$$\frac{\partial p(r, \tau)}{\partial \tau} + v^*(\tau) \frac{\partial p(r, \tau)}{\partial r}$$
$$= -\mu(r)[v(\tau)\, p^*(r, \tau) + v^*(\tau)\, p(r, \tau)] - v(\tau) \frac{\partial p^*(r, \tau)}{\partial r},$$

$$p(r, 0) = 0,$$

$$v(\tau)\, p^*(0, \tau) + v^*(\tau)\, p(0, \tau)$$
$$= v(\tau)\, \beta^*(\tau) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, \tau)\, dr$$
$$+ v^*(\tau)\, \beta^*(\tau) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, \tau)\, dr. \tag{25}$$

For any $f \in K_2^*$, $f = f_{11} + f_{12}$, $f_{1i} \in K_{1i}^*$, $i = 1, 2$,

$$f_{12}(p, v) = \int_0^{r_m} \alpha(r)\, p(r, 1)\, dr, \qquad \alpha(r) \in L^2(0, 1). \tag{26}$$

By the Dubovitskii–Milyutin Theorem, there exist functionals $f_i \in K_i^*$, $i = 0, 1, 2$, not all identically zero such that

$$f_0 + f_1 + f_{11} + f_{12} = 0. \tag{27}$$

In particular for any (p, v) satisfying (25), $f_{11}(p, v) = 0$, and so

$$f_1(p, v) = -f_0(p, v) - f_{12}(p, v)$$
$$= \lambda_0 \int_0^1 \int_m^r \left[\frac{\partial L(p^*(r, \tau), \beta^*(\tau))}{\partial p} v^*(\tau)\, p(r, \tau) \right.$$
$$\left. + L(p^*, \beta^*)\, v(\tau) \right] dr\, d\tau - \int_0^{r_m} \alpha(r)\, p(r, 1)\, dr, \tag{28}$$

where the solution of (25) is considered as that of the integral equation

$$\int_0^r p(s, \tau)\, ds + \int_0^\tau \left[[v(\xi)\, p^*(r, \xi) + v^*(\xi)\, p(r, \xi)] \right.$$
$$\left. - \beta^*(\xi) \int_{r_1}^{r_2} k(r)\, h(r) [v(\xi)\, p^*(r, \xi) + v^*(\xi)\, p(r, \xi)]\, dr \right] d\xi$$
$$+ \int_0^r \int_0^\tau \mu(s) [v(\xi)\, p^*(s, \xi) + v^*(\xi)\, p(s, \xi)]\, ds\, d\xi = 0. \tag{29}$$

Define the adjoint equation

$$\frac{\partial q(r,\tau)}{\partial r}+v^*(\tau)\,\frac{\partial q(r,\tau)}{\partial \tau}$$
$$=v^*(\tau)\,\mu(r)\,q(r,\tau)-\beta^*(\tau)\,k(r)\,h(r)\,q(\tau)+\lambda_0\,\frac{\partial L(p^*,\beta^*)}{\partial p}$$
$$q(r,1)=\alpha(r)$$
$$v^*(\tau)\,q(0,\tau)=v^*(\tau)\,q(\tau) \tag{30}$$

and

$$\int_r^{r_m}\hat{q}(s,\tau)\,ds=q(r,\tau). \tag{31}$$

As in [1], we have

LEMMA 1. *The solution of Eq.* (25) *and the solution of Eqs.* (30), (31) *have the relation*

$$\lambda_0\int_0^1\int_0^{r_m}\left[\frac{\partial L(p^*(r,\tau),\beta^*(\tau))}{\partial p}\,v^*(\tau)\,p(r,\tau)\right.$$
$$\left.+L(p^*,\beta^*)\,v(\tau)\right]dr\,d\tau-\int_0^{r_m}\alpha(r)\,p(r,1)\,dr$$
$$=\int_0^1\left[\int_0^{r_m}p^*(r,\tau)\,\hat{q}(r,\tau)+\beta^*(\tau)\int_r^{r_2}k(r)\,h(r)\,p^*(r,\tau)\,q(\tau)\right.$$
$$\left.+\int_0^{r_m}\mu(r)\,p^*(r,\tau)\,q(r,\tau)\,dr\right]v(\tau)\,d\tau. \tag{32}$$

Lemma 1 together with (28) and (18) implies that

$$\left[\int_0^{r_m}p^*(r,\tau)\,\hat{q}(r,\tau)\,dr\,d\tau+\beta^*(\tau)\int_{r_1}^{r_2}k(r)\,h(r)\,p^*(r,\tau)\,q(\tau)\,dr\,d\tau\right.$$
$$\left.+\int_0^{r_m}\mu(r)\,p^*(r,\tau)\,q(r,\tau)\,dr+\lambda_0\int_0^{r_m}L(p^*,\beta^*)\,dr\right][v-v^*(\tau)]\geqslant 0$$
$$\text{for all } v\geqslant 0. \tag{33}$$

It follows from (33) that

$$\int_0^{r_m} p^*(r,\tau)\,\hat{q}(r,\tau)+\beta^*(\tau)\int_r^{r_2} k(r)\,h(r)\,p^*(r,\tau)\,q(\tau)\,dr$$
$$+\int_0^{r_m} \mu(r)\,p^*(r,\tau)\,q(r,\tau)\,dr+\lambda_0\int_0^{r_m} L(p^*,\beta^*)\,dr=0, \qquad \forall\tau\in S_1, \quad (34)$$

$$\int_0^{r_m} p^*(r,\tau)\,\hat{q}(r,\tau)+\beta^*(\tau)\int_r^{r_2} k(r)\,h(r)\,p^*(r,\tau)\,q(\tau)\,dr$$
$$+\int_0^{r_m} \mu(r)\,p^*(r,\tau)\,q(r,\tau)\,dr+\lambda_0\int_0^{r_m} L(p^*,\beta^*)\,dr\geqslant 0, \qquad \forall\tau\in S_2. \quad (35)$$

We say that λ_0 and $\alpha(r)$ cannot be both zero, since otherwise, $f_0=0$, $q(s,\tau)=0$, $f_{12}=0$, $f_1=0$ and hence $f_{11}=0$. This contradicts the Dubovitskii–Milyutin Theorem. Furthermore, if $K_0=\varnothing$, take $\lambda_0=1$, $\alpha(r)=0$, then (32) implies (33) and hence (34) and (35) are valid. Finally, if Eq. (30) has a nonzero solution $q(r,\tau)$ such that

$$\int_0^{r_m} p^*(r,\tau)\,\hat{q}(r,\tau)+\beta^*(\tau)\int_r^{r_2} k(r)\,h(r)\,p^*(r,\tau)\,q(\tau)\,dr$$
$$+\int_0^{r_m} \mu(r)\,p^*(r,\tau)\,q(r,\tau)\,dr=0, \quad (36)$$

then take $\lambda_0=0$, and (33) is also valid. On the other hand, for any nonzero solution of (30)

$$\int_0^{r_m} p^*(r,\tau)\,\hat{q}(r,\tau)+\beta^*(\tau)\int_r^{r_2} k(r)\,h(r)\,p^*(r,\tau)\,q(\tau)\,dr$$
$$+\int_0^{r_m} \mu(r)\,p^*(r,\tau)\,q(r,\tau)\,dr\neq 0. \quad (37)$$

We call this situation the nondegenerate case, since here the linearized system must be controllable. This is because otherwise there exists a $\alpha(r)\in L^2(0,r_m)$ such that $\int_0^{r_m}\alpha(r)\,p(r,1)\,dr=0$, $\alpha(r)\neq 0$, and taking $\lambda_0=0$, we have a contradiction to (36). Hence, no matter what happened, (33) and (35) are always valid.

Define $q(r,t)=q(r,\tau(t))$, $\hat{q}(r,t)=\hat{q}(r,\tau(t))$, $q(t)=q(0,\tau(t))$, then (34) can be written as

$$\int_0^{r_m} p^*(r,t)\,\hat{q}(r,t)+\beta^*(t)\int_r^{r_2} k(r)\,h(r)\,p^*(r,t)\,q(t)\,dr$$
$$+\int_0^{r_m} \mu(r)\,p^*(r,t)\,q(r,t)\,dr+\lambda_0\int_0^{r_m} L(p^*,\beta^*)\,dr=0,$$
$$\text{for all } t\in[0,t_1] \text{ a.e.} \quad (38)$$

Choose S_1 to be a perfect nowhere dense subset of [0, 1] (see [2]) and define

$$v^*(\tau)=\begin{cases} t_1/\mu(s_1), & \tau\in S_1 \\ 0, & \tau\in S_2=[0,1]\backslash S_2. \end{cases} \tag{39}$$

Now, analysing the condition (35) as in [2], we can define $\beta^*(\tau)$ on S_2 and get (with the same notation as before)

$$\int_0^{r_m} p^*(r,t)\,\hat{q}(r,t)+\beta\int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r,t)\,q(t)\,dr$$
$$+\int_0^{r_m}\mu(r)\,p^*(r,t)\,q(r,t)\,dr+\lambda_0\int_0^{r_m} L(p^*,\beta)\,dr\geqslant 0,\qquad \forall\beta\in M, \tag{40}$$

for all $t\in[0,t_1]$. We have thus proved the following

THEOREM 1 (Maximum Principle). *Under the conditions on L mentioned in the beginning of this paper, and letting* (β^*, p^*, t_1) *solve Problem* (P), *then there exist* $q(r,t)$, $\lambda_0\geqslant 0$, *not both zero, such that*

$$\int_0^{r_m} p^*(r,t)\,\hat{q}(r,t)+\beta^*(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r,t)\,q(t)\,dr$$
$$+\int_0^{r_m}\mu(r)\,p^*(r,t)\,q(r,t)\,dr+\lambda_0\int_0^{r_m} L(p^*,\beta^*)\,dr=0,\ \forall t\in[0,t_1]\ a.e.$$

$$\int_0^{r_m} p^*(r,t)\,\hat{q}(r,t)+\beta\int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r,t)\,q(t)\,dr$$
$$+\int_0^{r_m}\mu(r)\,p^*(r,t)\,q(r,t)\,dr+\lambda_0\int_0^{r_m} L(p^*,\beta)\,dr\geqslant 0,$$
$$\forall\beta\in M.\quad t\in[0,t_1]\ a.e.,$$

where

$$\frac{\partial q(r,t)}{\partial r}+\frac{\partial q(r,t)}{\partial t}=\mu(r)\,q(r,t)-\beta^*(t)\,k(r)\,h(r)\,q(t)+\lambda_0\frac{\partial L(p^*,\beta^*)}{\partial p},$$
$$q(r,t_1)=\alpha(r),$$
$$q(0,t)=q(t),$$
$$q(r,t)=\int_r^{r_m}\hat{q}(s,t)\,ds. \tag{41}$$

Note. If the end point condition $p(r, t_1) = p^0(r)$ is imposed instead of

$$p(r, t_1) \in \{ p(r) \mid \| p(r) - p^0(r) \| \leqslant \varepsilon \} \tag{42}$$

then $\alpha(r)$ should be taken as $\alpha(r) = p^*(r, t_1) - p^0(r)$ and λ_0 can be set to 1.

COROLLARY 1. *If $L = 1$, then problem* (P) *is the time optimal control problem considered in* [1] *and the time optimal control satisfies the maximum principle*

$$\beta^*(t)\, H(t) = \max_{\beta \in M} \beta H(t), \qquad \forall t \in [0, t_1] \text{ a.e.}$$

$$H(t) = q(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, dr, \tag{43}$$

where t_1 is the minimum time. $q(t)$ is the solution of adjoint equation (41).

The result is the same as that of [1] but there the convexity assumption on M is not assumed.

6. SYSTEM WITH PHASE CONSTRAINTS

In this part, we consider the optimal control problem of a population system with phase constraints

$$\text{Problem (Q): Minimize } \hat{J}(\beta, p) = \int_0^T \int_0^{r_m} Q(p(r, t), \beta(t), t)\, dr\, dt \tag{44}$$

under the constraints

$$\begin{aligned}
&\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r)\, p(r, t), \qquad 0 < r < r_m, \quad t > 0,\\
&p(r, 0) = p_0(r), \qquad 0 \leqslant r \leqslant r_m,\\
&p(r, T) = p^0(r), \qquad 0 \leqslant r \leqslant r_m,\\
&p(0, t) = \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr, \qquad t \geqslant 0,\\
&\beta(t) \in [\beta_0, \beta_1] \qquad \text{for} \quad t \in [0, T] \text{ a.e.}\\
&\int_0^{r_m} G(p(r, t), t)\, dr \leqslant 0, \qquad t \geqslant 0,
\end{aligned} \tag{45}$$

in the class of

$$(p(r,t),\beta(t))\in X=C(0,T;L^2(0,r_m))\times L^\infty(0,T). \tag{46}$$

The time T is fixed.

Define

$$Q_1=\{(p(r,t),\beta(t))\in X\,|\,\beta(t)\in[\beta_0,\beta_1],\ t\in[0,T]\text{ a.e.}\} \tag{47}$$

$$Q_2=\Big\{(p(r,t),\beta(t))\in X\,|\,p_t=-\mu p,$$

$$p(0,t)=\beta(t)\int_{r_1}^{r_2}k(r)\,h(r)\,p(r,t)\,dr,\ p(r,0)=p_0(r),\ p(r,T)=p^0(r)\Big\} \tag{48}$$

$$Q_3=\left\{(p(r,t),\beta(t))\in X\,\middle|\,\int_0^{r_m}G(p(r,t),t)\,dr\leqslant 0\right\}. \tag{49}$$

Then Problem (Q) is equivalent to finding $(p^*,\beta^*)\in Q_1\cap Q_2\cap Q_3$ such that

$$\hat{J}(\beta^*,p^*)=\min_{(p,\beta)\in Q_1\cap Q_2\cap Q_3}\hat{J}(\beta,p). \tag{50}$$

This is a minimum problem formed by the inequality constraints Q_1, Q_3 and the equality Ω_2. We can use again the general theory of Dubovitskii–Milyutin for the extremum problem.

We had already investigated the corresponding cones of Q_1 and Q_2 of the Dubovitskii–Milyutin Theorem. Now we need only to consider constraint Q_3. Notice that Q_3 can be written as

$$Q_3=\{(p(r,t),\beta(t))\in X\mid F(p)\leqslant 0\}, \tag{51}$$

where $F(p)=\max_{0\leqslant t\leqslant 1}\int_0^{r_m}G(p(r,t),t)\,dr$ and assume

(1) $\int_0^{r_m}G(p(r),t)\,dr$ is a continuous functional on $L^2(0,r_m)\times[0,\infty]$;

(2) $\int_0^{r_m}G(p_0(r),0)\,dr<0$, $\int_0^{r_m}G(p^0(r),T)\,dr<0$;

(3) $\int_0^{r_m}G'_p(p(r),t)\,dr$ is also continuous on $L^2(0,r_m)\times[0,\infty)$ and $\int_0^{r_m}G'_p(p(r),t)\,dr\neq 0$ if $\int_0^{r_m}G(p(r),t)\,dr=0$.

Let (β^*,p^*) solve Problem (Q), then we consider $F(p^*)=0$, since otherwise, the feasible direction cone K_3 of Q_3 at (β^*,p^*) is the whole space, i.e., $K_3=X$. So $Q_3=\{(p(r,t),\beta(t))\in X\mid F(p)\leqslant F(p^*)\}$. Applying arguments as in [2] we can prove that

LEMMA 2. *$F(p)$ is differentiable at any point in any direction and*

$$F'(\hat{p}, p) = \max_{t \in S} \int_0^{r_m} G_p'(\hat{p}(r, t), t)\, p(r, t)\, dr, \tag{52}$$

where $S = \{t \in [0, T] \mid \int_0^{r_m} (\hat{p}(r, t), t)\, dr = F(\hat{p})\}$. Furthermore $F(p)$ satisfies a Lipschitz condition in any ball.

Notice that $F'(p^*, G_p'(p^*, t)) < 0$, we know that [2]

$$K_3 = \{(p, \beta) \in X \mid F'(p^*, p) < 0\}. \tag{53}$$

Define the linear operator $A: X \to C[0, T]$ by

$$Ap(r, t) = -\int_0^{r_m} G_p'(p^*(r, t), t)\, p(r, t)\, dr \tag{54}$$

and

$$K = \{y(t) \in C[0, T] \mid y(t) \geqslant 0, \forall t \in S\}$$

then $K_3 = \{p(r, t) \in X \mid Ap \in K\}$. Since $A(-G_p'(p^*(r, t), t)) \in \mathring{K}$, so $K_3^* = A^* K^*$; i.e., for any $f \in K_3^*$, there exists a measure $dm(t)$, nonnegative and with support on S, such that

$$\begin{aligned} f(p(r, t)) &= \int_0^T Ap(r, t)\, dm(t) = \int_S Ap(r, t)\, dm(t) \\ &= -\int_S \int_0^{r_m} G_p'(p^*(r, t), t)\, p(r, t)\, dr\, dm(t). \end{aligned} \tag{55}$$

Based on the previous results, there exist $\lambda_0 \geqslant 0$, $\alpha(r) \in L^2(0, r_m)$ such that

$$\begin{aligned} f_1(p, \beta) = \lambda_0 &\int_0^T \int_0^{r_m} \left[\frac{\partial Q(p^*, \beta^*, t)}{\partial p}\, p(r, t) + \frac{\partial Q(p^*, \beta^*, t)}{\partial \beta}\, \beta(t)\right] dr\, dt \\ &- \int_0^{r_m} p(r, T)\, \alpha(r)\, dr + \int_0^T \int_0^{r_m} G_p'(p^*(r, t), t)\, p(t, t)\, dr\, dm(t), \end{aligned} \tag{56}$$

where (p, β) satisfies

$$\frac{\partial p(r,t)}{\partial t}+\frac{\partial p(r,t)}{\partial r}=-\mu(r)\,p(r,t),$$

$$p(r,0)=0,$$

$$p(0,t)=\beta^*(t)\int_{r_1}^{r_2}k(r)\,h(r)\,p(r,t)\,dr$$

$$+\beta(t)\int_{r_1}^{r_2}k(r)\,h(r)\,p^*(r,t)\,dr, \qquad (57)$$

with the assumption that the decreasing direction cone of J at (p^*, β^*) is not empty and system (57) is controllable.

Define the adjoint system

$$\frac{\partial q(r,t)}{\partial r}+\frac{\partial q(r,t)}{\partial t}=\mu(r)\,q(r,t)-\beta^*(t)\,k(r)\,h(r)\,q(t)$$

$$+\lambda_0\frac{\partial Q(p^*,\beta^*,t)}{\partial p}+G_p'(p^*(r,t),t)\frac{dm(t)}{dt}$$

$$q(r,T)=\alpha(r),$$

$$q(0,t)=q(t). \qquad (58)$$

The solution of Eq. (58) should be considered as that of the integral equation

$$-\int_0^r q(s,t)\,ds$$

$$=-\int_0^r \alpha(s)\,ds-\int_0^r\int_t^T [q(s,\tau)-q(\tau)]\,ds\,d\tau+\int_0^r\int_t^T \mu(s)\,q(s,\tau)\,ds\,d\tau$$

$$=-\int_0^r k(s)\,h(s)\,ds\int_t^T \beta^*(\tau)\,q(\tau)\,d\tau+\lambda_0\int_0^r\int_t^T\frac{\partial Q}{\partial p}\,ds\,d\tau$$

$$+\int_0^{r_m}\int_t^T G_p'(p^*(s,\tau),\tau)\,dm(\tau)\,dr. \qquad (59)$$

As before, we have

LEMMA 3. *The solution of Eq. (57) and the adjoint equation has the relation*

$$\lambda_0 \int_0^T \int_0^{r_m} \left[\frac{\partial Q(p^*, \beta^*, t)}{\partial p} p(r, t) + \frac{\partial Q(p^*, \beta^*, t)}{\partial \beta} \beta(t) \right] dr\, dt$$

$$- \int_0^{r_m} p(r, T)\, \alpha(r)\, dr + \int_0^T \int_0^{r_m} G_p'(p^*(r, t), t)\, p(r, t)\, dr\, dm(t)$$

$$= \int_0^T \left[\int_0^{r_m} \left[\lambda_0 \frac{\partial Q(p^*, \beta^*, t)}{\partial \in \beta} - q(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, dr \right] \beta(t)\, dt \right]. \tag{60}$$

Same reason as before, whether or not the decreasing direction cone of J at (p^*, β^*) is empty and the system (57) is controllable, we always have

THEOREM 2 (Maximum Principle). *Let (p^*, β^*) solve Problem* (Q), *then there exist $\lambda_0 \geqslant 0$, $q(t)$ not both zero, such that*

$$\int_0^{r_m} \left[\lambda_0 \frac{\partial Q(p^*, \beta^*, t)}{\partial \beta} - q(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, dr \right] [\beta - \beta^*(t)] \geqslant 0,$$
$$\forall t \in [0, T] \text{ a.e.} \tag{61}$$

We can also consider the free final time problem with phase constraints

$$\text{Problem (W): Minimize } \bar{J}(\beta, p) = \int_0^{t_1} \int_0^{r_m} W(p(r, t), \beta(t), t)\, dr\, dt$$

under the constraints

$$\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r)\, p(r, t), \qquad 0 < r < r_m,\ t > 0,$$

$$p(r, 0) = p_0(r), \qquad 0 \leqslant r \leqslant r_m,$$

$$p(r, t_1) = p^0(r), \qquad 0 \leqslant r \leqslant r_m,$$

$$p(0, t) = \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr, \qquad t \geqslant 0,$$

$$\beta(t) \in M, \qquad \text{for} \quad t \in [0, t_1] \text{ a.e.}$$

$$\int_0^{r_m} G(p(r, t), t)\, dr \leqslant 0, \qquad t \geqslant 0, \tag{62}$$

in the class of

$$(p(r, t), \beta(t)) \in X = C(0, t_1; L^2(0, r_m)) \times L^\infty(0, t_1). \tag{63}$$

The time t_1 is free.

Following the same lines of reasoning in Section 5, we can prove

THEOREM 3. *Let* (p^*, β^*, t_1) *solve Problem* (W), *then there exist* λ_0, $\alpha(r) \in L^2(0, r_m)$ *with support on* $S = \{t \in [0, T] \mid \int_0^{r_m} G(\hat{p}(r, t), t)\, dr = F(\hat{p})\}$ *and a nonnegative measure* $dm(t)$ *such that*

$$\int_0^{r_m} p^*(r, t)\, \hat{q}(r, t) + \beta^*(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, q(t)\, dr$$

$$+ \int_0^{r_m} \mu(r)\, p^*(r, t)\, q(r, t)\, dr + \lambda_0 \int_0^{r_m} W(p^*, \beta^*)\, dr = 0,$$

$$\forall t \in [0, t_1] \text{ a.e.} \qquad (64)$$

$$\int_0^{r_m} p^*(r, t)\, \hat{q}(r, t) + \beta \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, q(t)\, dr$$

$$+ \int_0^{r_m} \mu(r)\, p^*(r, t)\, q(r, t)\, dr + \lambda_0 \int_0^{r_m} W(p^*, \beta)\, dr \geqslant 0,$$

$$\forall t \in [0, t_1] \text{ a.e.}, \qquad (65)$$

where

$$\frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} = \mu(r)\, q(r, t) - \beta^*(t)\, k(r)\, h(r)\, q(t)$$

$$+ \lambda_0 \frac{\partial W(p^*, \beta^*)}{\partial p}, G'_p(p^*(r, t), t) \frac{dm(t)}{dt}$$

$$q(r, t_1) = \alpha(r),$$

$$q(0, t) = q(t) \qquad (66)$$

$$q(r, t) = \textstyle\int_r^{r_m} \hat{q}(s, t)\, ds.$$

7. MINI-MAX PROBLEMS

The mini-max control problem of a population control system can be stated as

$$\text{Problem (Y): Minimize } F(p) = \max_{0 \leqslant t \leqslant t_1} \int_0^{r_m} G(p(r, t), t)\, dr\, dt \qquad (67)$$

with respect to $(p(r, t), \beta(t)) \in X$ and t_1 under the constraints

$$\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r)\, p(r, t), \qquad 0 < r < r_m,\ t > 0,$$

$$p(r, 0) = p_0(r), \qquad 0 \leqslant r \leqslant r_m,$$

$$p(r, t_1) = p^0(r), \qquad 0 \leqslant r \leqslant r_m,$$

$$p(0, t) = \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr, \qquad t \geqslant 0,$$

$$\beta(t) \in M, \qquad \text{for} \quad t \in [0, t_1] \text{ a.e.} \tag{68}$$

We only state the results since the proof is similar.

THEOREM 4. *Let $\int_0^{r_m} G(p(r), t)\, dr$ be continuously differentiable with respect to $p(r)$, $\int_0^{r_m} G'_p(p(r), t) \neq 0$ when $G(p(r), t) \neq 0$. Let (p^*, β^*, t_1) solve Problem* (Y), *then there exist $q(r, t)$, $\alpha(r) \in L^2(0, r_m)$ and a nonnegative measure $dm(t)$ with support on the set*

$$S = \left\{ t \in [0, t_1] \,\middle|\, \int_0^{r_m} G(p^*(r, t), t)\, dr = \max_{0 \leqslant t \leqslant t_1} \int_0^{r_m} G(p^*(r, t), t)\, dr\, dt \right\}$$

such that

$$\int_0^{r_m} p^*(r, t)\, \hat{q}(r, t) + \beta^*(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, q(t)\, dr$$
$$+ \int_0^{r_m} \mu(r)\, p^*(r, t)\, q(r, t)\, dr, \qquad \forall t \in [0, t_1] \text{ a.e.} \tag{69}$$

$$\int_0^{r_m} p^*(r, t)\, \hat{q}(r, t) + \beta \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, q(t)\, dr$$
$$+ \int_0^{r_m} \mu(r)\, p^*(r, t)\, q(r, t)\, dr \geqslant 0, \qquad \forall \beta \in M,\ t \in [0, t_1] \text{ a.e.}, \tag{70}$$

where $q(r, t)$ is the solution of the adjoint equation

$$\frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} = \mu(r)\, q(r, t) - \beta^*(t)\, k(r)\, h(r)\, q(t) + G'_p(p^*) \frac{dm(t)}{dt}$$

$$q(r, t_1) = \alpha(r),$$

$$q(0, t) = q(t) \tag{71}$$

$$q(r, t) = \int_r^{r_m} \hat{q}(s, t)\, ds.$$

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